Characterizing Streaming Decidability of CSPs via Non-Redundancy

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Thanks to NSERC Discovery grant, NSF award and ECF supplement, and University of Michigan Rackham Conference Travel Grant

Constraint Satisfaction Problems

A fixed local predicate/relation: $R \subseteq D^k$

Variables: $x_1, \ldots, x_n$

Constraints: $R(x_{i_1}, \ldots, x_{i_k})$

An assignment is satisfying if it satisfies every constraint.

Examples

$\neq$ → graph coloring / bipartiteness

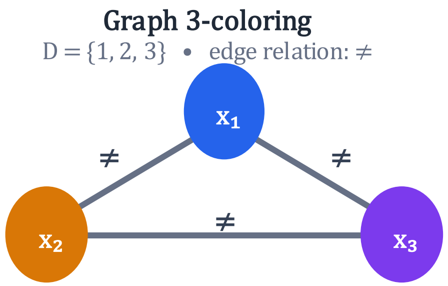
NAE_k → hypergraph 2-coloring

OR_k → k-SAT

$\oplus_k$ → linear equations

Constraint languages: more generally, allow a finite family $\Gamma = \{R_1, \ldots, R_t\}$.
Example: Unique Games: each permutation π gives the relation $R_\pi = \{(a, \pi(a)) : a \in [q]\}$.

Our results hold for finite Γ; for this talk, we fix one relation R.



The streaming satisfiability problem

INPUT: A CSP INSTANCE

A fixed local predicate/relation: $R \subseteq D^k$

Variables: $x_1, \dots, x_n$

Constraints arrive one at a time

C_1	C_2	C_3	$\cdots$	C_m

time

Each constraint applies R to some k variables.

one pass • minimize space • time unrestricted

ALGORITHM'S WORKING MEMORY

small memory

M_t

a summary of everything seen so far

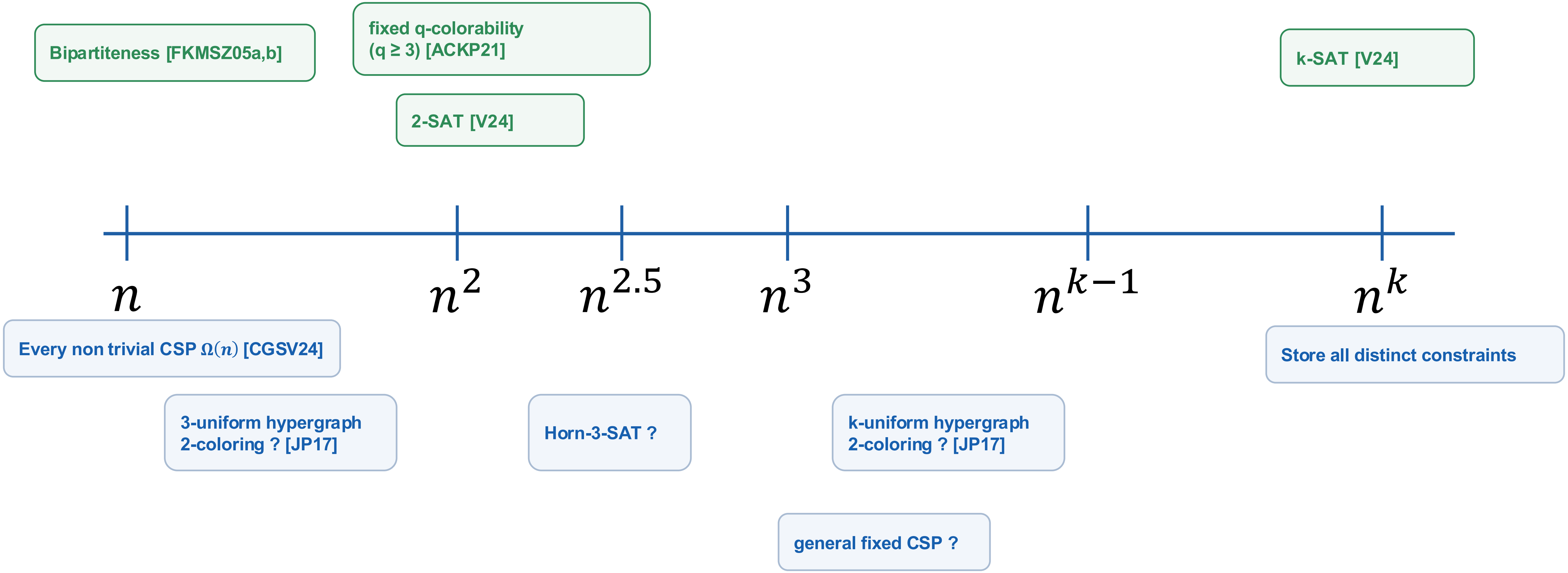
As each constraint arrives:

$$M_t = Update(M_{t-1}, C_t)$$

Goal: decide whether some assignment satisfies every constraint.

For a fixed relation R, how much memory is necessary and sufficient?

Motivation: different predicates give different exponents



What property of R determines the exponent?

Main result: the answer is non-redundancy

For every fixed relation R, the optimal one-pass streaming space is

$$\widetilde{\Theta}(\mathrm{NRD}_n(R))$$

This gives tight bounds for the cases left open on the previous slide:

3-uniform hypergraph 2-coloring	$\tilde{\Theta}(n^2)$
k-uniform hypergraph 2-coloring	$\tilde{\Theta}(n^{k-1})$
Horn-3-SAT	$\tilde{\Theta}(n^3)$

So what is non-redundancy?

Non-redundancy

Definition

An instance is non-redundant if removing any constraint changes the set of satisfying assignments.

$\boxed{\mathrm{NRD}_n(R) :=}$ largest size of a non-redundant instance on n variables.

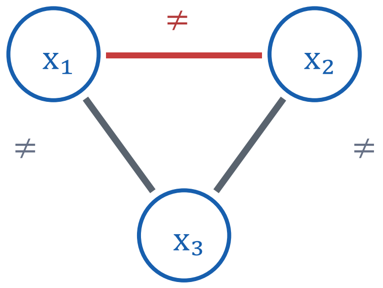
[BCK20, C22]

Witness Assignment

For every constraint in a non-redundant instance, there is an assignment that satisfies all the other constraints, but violates this one.

This is exactly the perspective we will use later in the lower bound.

Bipartiteness/XOR example



$(x_1 = 0,\ \ x_2 = 0,\ \ x_3 = 1)$

= Witness assignment for edge (x_1, x_2)

$$\mathrm{NRD}_n(\mathrm{XOR}) = \Omega(n)$$

Non-redundancy

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2-SAT example

Put one clause on every unordered pair {i,j}:

$$C_{ij} = x_i \vee x_j$$

$$\binom{n}{2} = \Theta(n^2)$$

Witness for C_{ij}:

$$x_i = x_j = 0, \qquad x_\ell = 1 \; (\ell \neq i,j)$$

Every other clause contains a variable set to 1.
Only C_{ij} is violated.

$$\mathrm{NRD}_n(2\text{-SAT}) = \Theta(n^2)$$

For every fixed R, the optimal one-pass streaming space is $\widetilde{\Theta}(\mathrm{NRD}_n(R))$

Upper bound: keep only a non-redundant subinstance

Maintain a subinstance with exactly the same satisfying assignments as all constraints seen so far.

current subinstance I

C_1 $\cancel{C_2}$ C_3 C_4 $+$ C

new constraint

$\downarrow$

Delete constraints one by one as long as it doesn't change the satisfying assignments.
An old constraint may become redundant.
Redundancy checks may be expensive; time is unrestricted in this model.

$\downarrow$

I' C_1 C_3 C_4 C non-redundant

Since I′ is non-redundant,

$$|I'| \leq \mathrm{NRD}_n(R)$$

Each constraint takes $O(\log n)$ bits to store.

$$\boxed{O(\mathrm{NRD}_n(R)\ \log n)\text{ space}}$$

Streaming lower bounds from INDEX

One-way communication INDEX

Alice	one message →	Bob
$x \in \{0,1\}^N$		x_i?

Streaming sat CSP(R)

Alice: $C_1, \ldots, C_N$ → (s)-bit state → Bob: $C'_1, \ldots, C'_m$

Any randomized one-way protocol requires $\Omega(N)$ bits. [KN97]

$$s\text{−space streaming CSP}(R) \;\Rightarrow\; s\text{−bit one−way communication for INDEX}$$

So: encode NRD-bit INDEX into streaming satisfiability

$\Rightarrow \Omega(\mathrm{NRD})$ space.

2-SAT encodes INDEX on all variable pairs

$N = \binom{n}{2}$

INDEX bits are indexed by unordered pairs {a,b}.

Alice

For every pair {a,b}:

$$z_{a,b} = 1 \Rightarrow \text{stream } C_{a,b} = (x_a \lor x_b)$$

→ s-bit state →

Bob receives (i,j)

Append clauses:

$$(\neg x_i \lor \neg x_i), \qquad (\neg x_j \lor \neg x_j)$$

forces $x_i = x_j = 0$

$$z_{i,j} = 1 \Rightarrow C_{i,j} \text{ present} \Rightarrow \text{UNSAT}$$

$$z_{i,j} = 0 \Rightarrow C_{i,j} \text{ absent} \Rightarrow \text{SAT}$$

SAT witness: $x_i = x_j = 0, \quad x_\ell = 1 \ (\ell \neq i, j)$ Exactly the private witness.

$$\Omega(N) = \Omega(n^2) \text{ space}$$

2-SAT: encoding INDEX on variable pairs

$N = \binom{n}{2}$ INDEX bits are indexed by unordered pairs {a,b}.

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For every pair {a,b}:

$z_{a,b} = 1 \Rightarrow$ stream $C_{a,b} = (x_a \lor x_b)$

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$z_{i,j} = 1 \Rightarrow C_{i,j}$ present $\Rightarrow$ UNSAT

$z_{i,j} = 0 \Rightarrow C_{i,j}$ absent $\Rightarrow$ SAT

SAT witness: $x_i = x_j = 0,\ x_\ell = 1\ (\ell \neq i,j)$

$\Omega(N) = \Omega(n^2)$ space

Generalizing the 2-SAT reduction

Take a largest non-redundant instance $I = \{C_1, \dots, C_N\}$, $N = \mathrm{NRD}_n(R)$.

For every C_i, choose a witness assignment a_i: a_i satisfies every other constraint, but violates C_i.

Alice

$z \in \{0,1\}^N$

For every j:

$z_j = 1 \Rightarrow$ stream C_j

Bob receives (i,j)

Append clauses:

force the witness assignment a_i

$z_i = 1 \Rightarrow C_i$ present $\Rightarrow$ UNSAT

because the forced witness a_i violates C_i.

$z_i = 0 \Rightarrow C_i$ absent $\Rightarrow$ SAT

because a_i satisfies every other C_j.

If Bob can force a_i, INDEX gives $\Omega(N) = \Omega(\mathrm{NRD}_n(R))$. Exactly the private witness.

But can Bob always force a_i using constraints from R?

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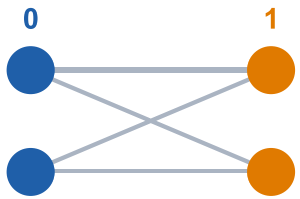
The obstacle: Bob may not be able to force one assignment

Example: Bipartitness / XOR $R(x,y) = [x \neq y]$

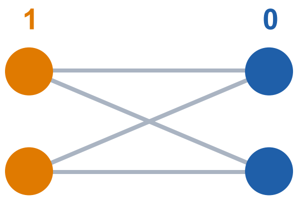
$a = (0,1,1,0, \ldots)$



flip every bit

$\leftrightarrow$

$\bar{a} = (1,0,0,1, \ldots)$



$$a \text{ satisfies } I \iff \bar{a} \text{ satisfies } I$$

for every XOR instance I.

$\downarrow$

~~force exactly a_i~~

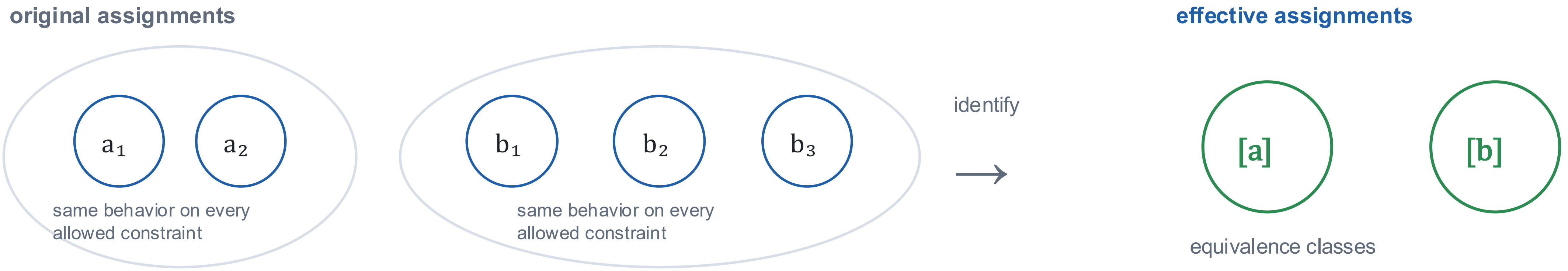
There may be no valid CSP instance whose unique satisfying assignment is a_i.

If R cannot distinguish two assignments, Bob should not need to distinguish them either.

What can R actually observe about an assignment?

The repair: identify indistinguishable assignments

If the CSP cannot distinguish two assignments, then they are effectively the same assignment.



Key fact

Bob can append a CSP instance whose satisfying assignments are exactly $[a_i]$.

Bob gets i $\longrightarrow$ $\mathrm{Sat}(I\_Bob) = [a_i]$ $\longrightarrow$ same INDEX reduction works $\longrightarrow$ $\Omega(\mathrm{NRD}_n(R))$ Lower bound

The One Ring: Non-Redundancy

Grazie!

On the 2027 postdoc market — happy to chat!

Streaming satisfiability

$\tilde{\Theta}(\mathrm{NRD})$

Approximate sparsification

preserve every assignment's value
using few weighted constraints

$\tilde{\Theta}(\mathrm{NRD})$

[BG25]

NRD

Streaming Approximate MinCSP

minimize number of
unsatisfied constraints

$\tilde{\Theta}(\mathrm{NRD})$

[BP**S**V26]

The same structural parameter appears across different CSP objectives. What other objectives?

Open problems

Understanding NRD

Known: symmetric Boolean CSPs classified through arity 4; almost all at arity 5.
[BGP26, **S**V26b, BGJLW26]

Simplest open cases

$$R = \{0,2,3\} \quad \text{or} \quad R = \{1,2,4\}$$

$$n^2 \lesssim \mathrm{NRD}_n(R) \lesssim n^3$$

What is the correct exponent?

Multiple passes

Partial p-pass bounds and tradeoffs are known.

$\Omega(\mathrm{NRD}(R)/p)$-type lower bounds?

and nontrivial space–pass tradeoffs for special cases such as 2-SAT

What is the right p-pass characterization for CSP satisfiability?