Bidimensionality

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Preliminaries

- Win\Win Approaches
- Vertex-subset problem and Edge-subset problem
- φ-Minimization and Maximization Problems
- OPT
- Contraction Closed and Minor closed
- Grid(t)
- Γ(t)
- Excluded grid theorem[1]
- Planar excluded grid theorem[2]
- Planar excluded grid theorem for edge contractions[3]

Preliminaries

Win\Win Approaches

→ win
→

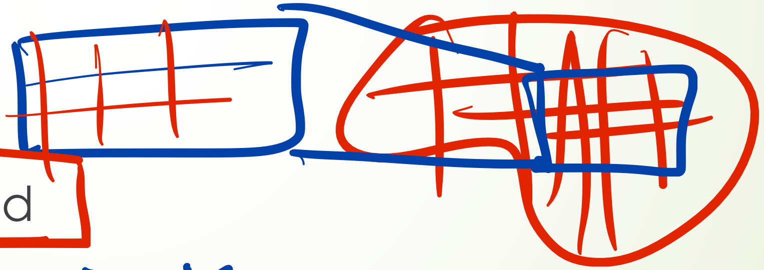
Tree width

$2^{O(tw)} n^{O(1)}$

VC
FVS
DS
EDGE?

Construct a good tree decomposition of the input graph

- Small Treewidth : Dynamic programming → win
- Large Treewidth : Existence of an obstacle
  - No-Instance ✓ →
  - Yes-Instance ✓
  - Reduction ✓
- Treewidth is minor-closed



$H \leq_m G$ then $tw(H) \leq tw(G)$

$tw(\ ) > k$

NO

→ YES

LP, LC

Vertex-subset problem and Edge-subset problem*

Input : Graph G and a parameter k

Goal : Decide whether there exists S and $\boxed{\varphi(G, S) = \text{true}}$ where :

- Vertex-subset problem

  $S \subseteq V(G)$ and $|S| \leq k$

- Edge-subset problem

  $S \subseteq E(G)$ and $|S| \leq k$

$\geq k$ $\Phi(G, S)$

$\geq k$

* for minimization

φ-Minimization and Maximization Problems*

Input : Graph G and a parameter k

Goal : Decide whether there exists a set $S \subseteq V(G)$ and $\varphi(G, S) = \text{true}$ and :

- **φ-Minimization**

  $|S| \leq k$

- **φ-Maximization**

  $|S| \geq k$

*for vertex-subset problem

Lets define OPT

- φ-Minimization problem Q

  $OPT_Q(G) = \min\{k : (G, k) \in Q\}$

  If for no k (G, k) ∈ Q, then $OPT_Q(G) = +\infty$

- φ-Maximization problem Q

  $OPT_Q(G) = \max\{k : (G, k) \in Q\}$

  If for no k (G, k) ∈ Q, then $OPT_Q(G) = -\infty$.

$\arg\min_k$ (G,k) φ(G,k) = true

null

$\arg\max_k$ (G,k) φ(G,k) = true

Contraction Closed and Minor closed

Vertex-subset problem Q

Q

- Contraction-closed
  - If for every contraction H of G, $OPT_Q(H) \leq OPT_Q(G)$ for all G → Q
- Minor-Closed
  - If for every minor H of G, $OPT_Q(H) \leq OPT_Q(G)$ for all G → Q

G → H

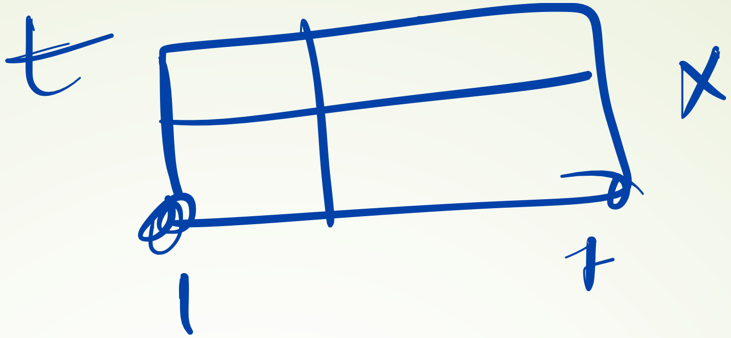
e contr

~~e delete~~

~~v delete~~

Preliminaries

Grid(t)

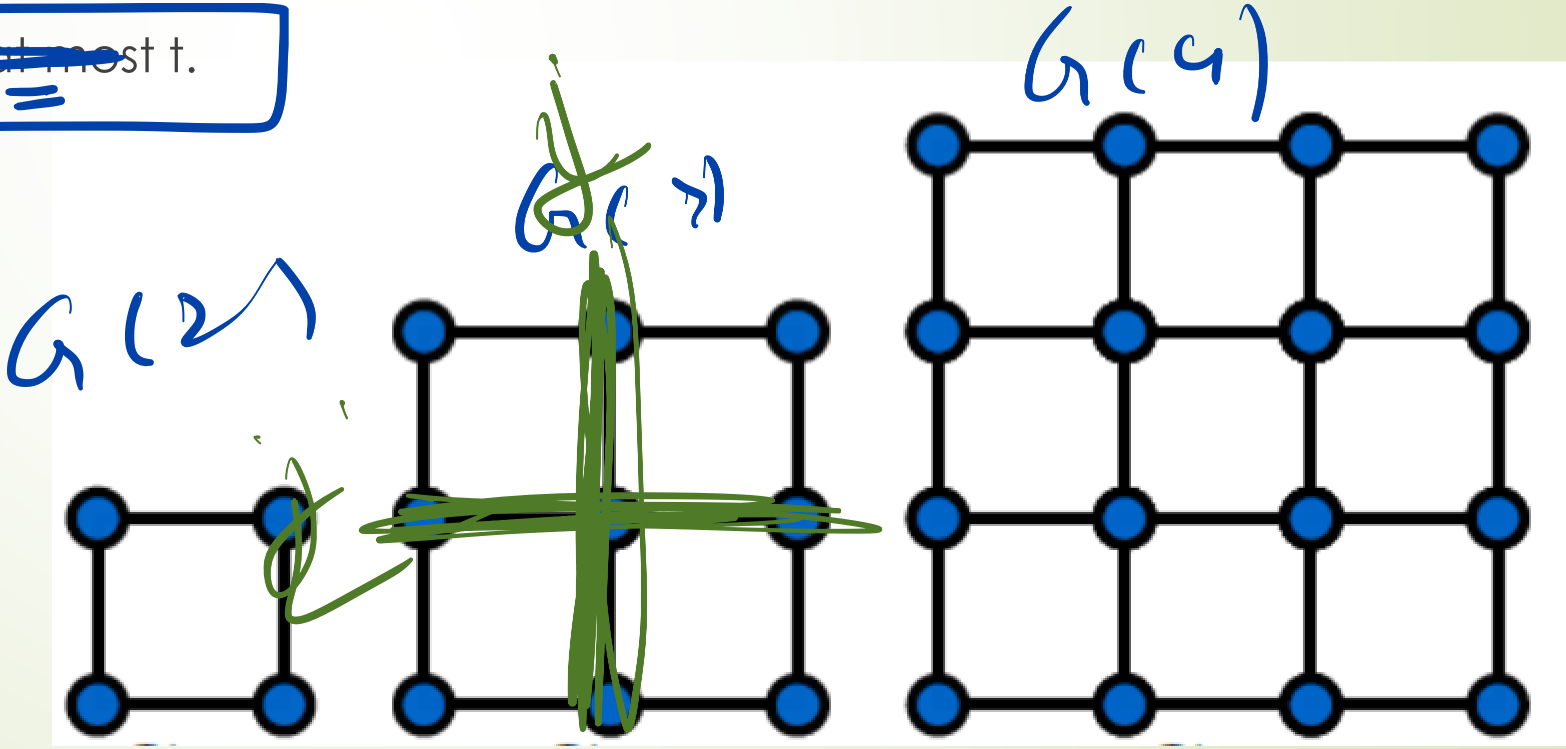


- Graph with vertex set {(x, y) : x, y ∈ {1,...,t}}. Thus t has exactly t^2 vertices.
- Two different vertices (x, y) and (x , y) are adjacent if and only if |x − x | + |y − y | = 1.
- $\boxed{\text{Treewidth is } \cancel{\text{at most}} = \text{ t.}}$

c_{ij} →

bramble

tw(Grid(t)) = t



Preliminaries

Γ(t)

minor-closed

contractly

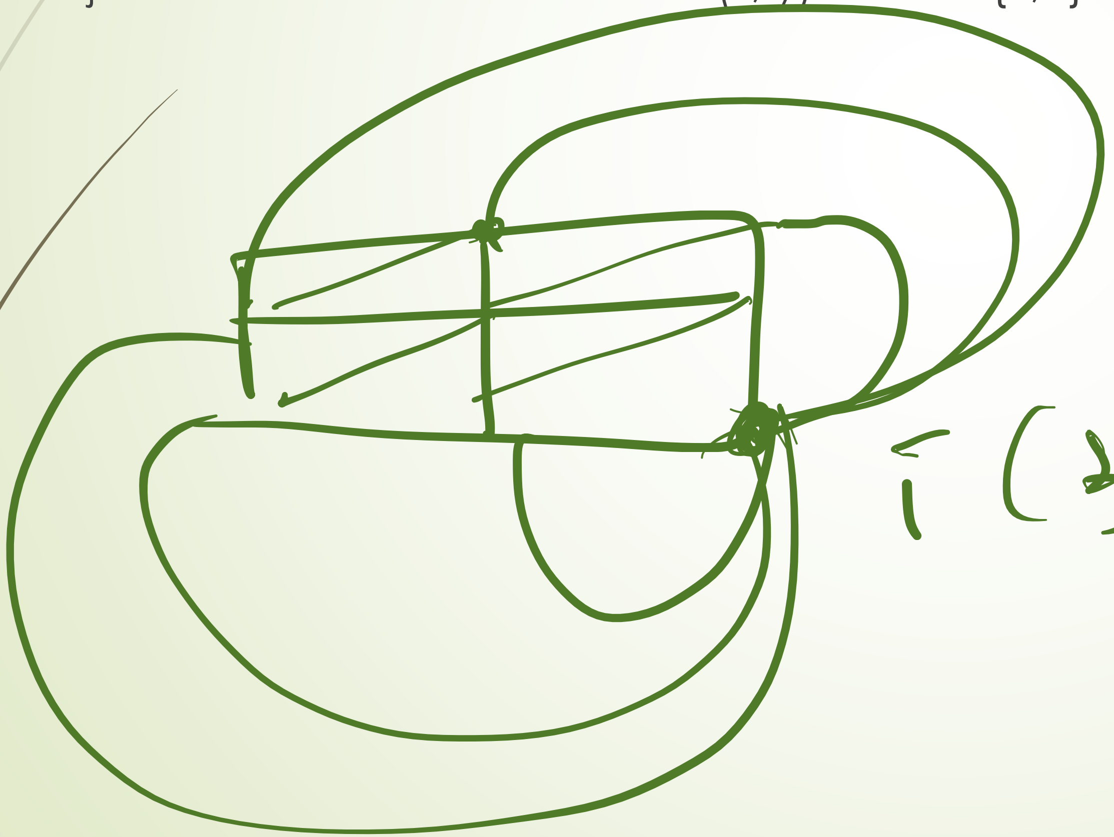
$\Gamma(t)$ contraction

Grid ℓ

$tw(\;) \geq t$

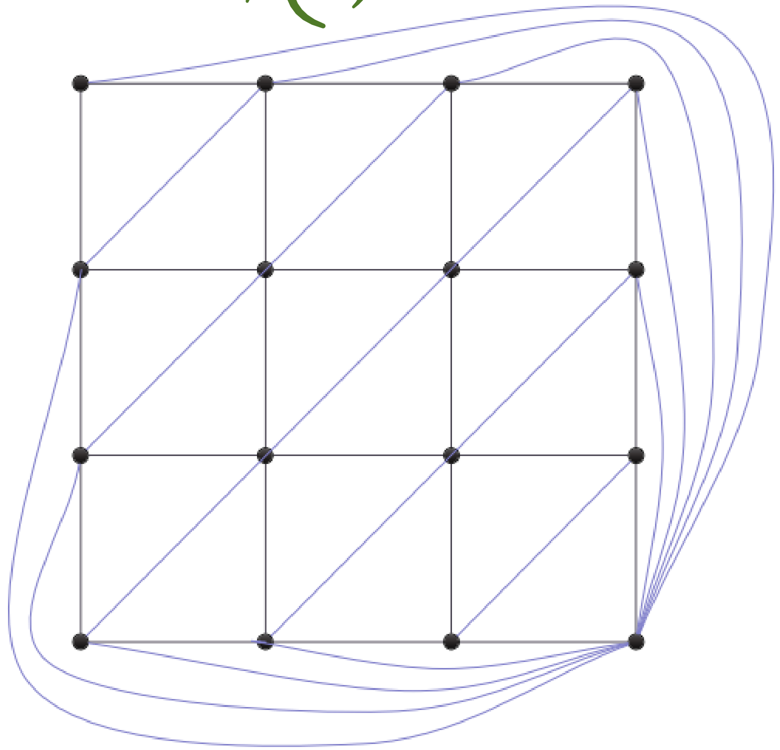
$tw(\Gamma(t)) \geq t$

- Obtained from the Grid(t) by
  - adding, for all $1 \leq x, y \leq t-1$, the edge $(x+1, y),(x, y+1)$, and additionally making vertex (t, t) adjacent to all the other vertices (x, y) with $x \in \{1, t\}$ or $y \in \{1, t\}$, i.e., to the whole border of t.



$\Gamma(3)$

$\Gamma(4)$



Excluded grid theorem[1]

- $\exists$ g(t) = $\boxed{O(t^{98+o(1)})}$ such that every graph of treewidth larger than g(t) contains Grid(t) as minor
- Robertson and Seymour g(t)= $2^{O(t^5)}$
- Randomized poly-time algorithm :
  - Either constructs a Grid(t) minor model
  - Or finds a tree decomposition of the input graph of width at most g(t)

Grid(t)

$\to \leq g(t)$

$tw(G) \geq g(t)$

Grid(t) minor

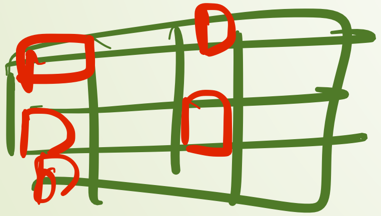
Lets Apply it on Cycle Packing

$\geq \underline{k}$ v.d.c

FPT Cycle Packing k

If G has treewidth larger than $g(t) = O(k^{49+o(1)})$:

- G contains a Grid(t) minor model for $t = 2\sqrt{k}$
- Then in this model one can find $(\sqrt{k})^2 \geq k$ vertex-disjoint cycles
- Run the approximation algorithm for parameter $g(t)$
  - If $tw(G) > g(t)$, then (G, k) is a yes-instance
  - Otherwise, we obtain a tree decomposition of G of width at most $4g(t)+4$



grid (t)

$\geq \frac{t^2}{4}$ v.d.c

$g(t)$

$\leq 4(g(t))+4$

YES

$tw > g(t)$

$\geq k$ v.d.c

Planar excluded grid theorem[2]

- Every planar graph G of treewidth at least 9t/2 contains Grid(t) as a minor
- For every ε > 0 there exists an $O(n^2)$ algorithm that, for a given n-vertex planar graph G and integer t
  - Either outputs a tree decomposition of G of width at most (9/2 + ε)t
  - Or constructs a minor model of Grid(t) in G

ε

$tw(G) \geq g(t) = 9\frac{t}{2} \rightarrow$ Grid(t)

$v \xrightarrow{\text{Grid}} S \subseteq G$

A Useful Corollary

- Treewidth of an n-vertex planar graph G is less than $\left\lceil \frac{9}{2}\sqrt{n+1} \right\rceil$
- For any ε > 0, a tree decomposition of G of width at most $\left\lceil \left(\frac{9}{2}+\varepsilon\right)\sqrt{n+1} \right\rceil$ can be constructed in $O(n^2)$ time.

$\text{Grid}(t) \to G$

t^2

$t^2 \leq n$

$t \leq \sqrt{n}$

$\text{Grid}(t+1) \to G$

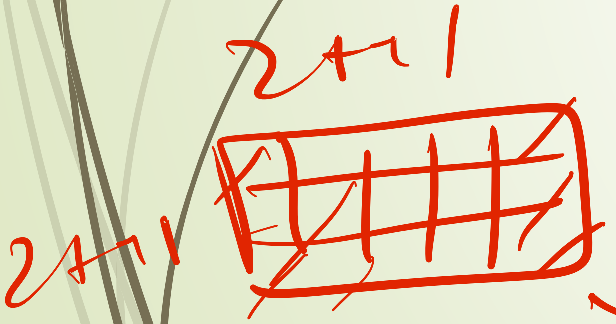
$\frac{9}{2}(t+1) > tw(G)$

$\left\lceil \frac{9}{2}(\sqrt{n}+1) \right\rceil > tw(G)$

Planar excluded grid theorem for edge contractions[3]

minor

- For every **connected** planar graph G and integer $t \geq 0$, if $tw(G) \geq 9t+5$ then G contains $\Gamma(t)$ as a contraction
- For every $\varepsilon > 0$ there exists an $O(n^2)$ algorithm that, given a connected planar n-vertex graph G and integer t
  - Either outputs a tree decomposition of G of width $(9 + \varepsilon)t + 5$
  - Or a set of edges whose contraction in G results in $\Gamma(t)$



$g(t) = 9t+5$

$\Gamma(t)$ ⊂ cut

$\Gamma(t)$

$t' = (2t+1)$

Grid(t') minor of G

Bidimensional Problem

A vertex-subset problem Q is bidimensional if

- Q is contraction-closed ⟷ minor-closedness
- $\exists$ constant $c > 0$ such that

$$OPT_Q(\Gamma(t)) \geq ct^2 \qquad \text{for every } t > 0$$

$$OPT_Q(\Gamma(t)) \geq ct^2$$

Thm : Parameter-treewidth bound[4]

Q be a bidimensional problem, then

- ∃ a constant a_Q such that for any connected planar graph G ✓

  $\boxed{tw(G) \leq a_Q \cdot \sqrt{OPT_Q(G)}}$

- ∃ a poly-time algorithm that for a given G constructs a tree decomposition of G of width at most $a_Q \cdot \sqrt{OPT_Q(G)}$ ✓

$f(\theta) \leftarrow \alpha \; OPT(G)$

tree decomp

$tw \leq \alpha \sqrt{OPT(G)}$

Thm : Parameter-treewidth bound[4]

Proof: $\varepsilon = 1$, $\alpha = \frac{24}{\sqrt{c}}$

$t' \rightarrow 1 \rightarrow$

$\vdots$

$\rightarrow$

tdec $(9+\varepsilon)t'+5$ $\Gamma(t')$

$O(n^2 \cdot t)$

$t' = t$

$t' < t$

$tw(G) \leq \alpha\sqrt{OPT(G)}$

BLACK BOX

Planar excluded grid theorem for edge contractions[3]

- For every **connected** planar graph G and integer t ≥ 0, if tw(G) ≥ 9t+5 then G contains Γ(t) as a contraction
- For every ε > 0 there exists an O(n²) algorithm that, given a connected planar n-vertex graph G and integer t
  - Either outputs a tree decomposition of G of width (9 + ε)t + 5
  - Or a set of edges whose contraction in G results in Γ(t)

BP – Theorem[5]

$t^{O(t)}\, n^{O(1)}$

Given : Bidimensional problem Q such that

Q can be solved in time $\boxed{2^{O(t)} \cdot n^{O(1)}}$

provided a tree decomposition of given G of width t

Then :

Q is solvable in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$ on **connected** planar graphs

FPT $2^{O(\sqrt{k})}\, n^{O(1)}$ $t^{O(\sqrt{t})}\, n^{O(1)}$

Connected p'o

$2^{O(t)}\, n^{O(1)} \longrightarrow$ FPT k

Bidim • 2

tw

BP – Theorem[5]

Proof:

- $tw(G) \le \alpha \sqrt{OPT(G)}$ $\to f(\alpha)$

BP elm

$tw(G) \ge \alpha\sqrt{k}$

$OPT(G) \ge k$

yes → max

no → min

win

$tw(G) < \alpha\sqrt{k}$

$\to \alpha\sqrt{K}$

$k^{O(\alpha\sqrt{k})}$ $n^{O(1)}$

$k^{O(\sqrt{k})}$ $n^{O(1)}$

APT

BLACK BOX

Thm : Parameter-treewidth bound[4]

Q be a bidimensional problem, then

- ∃ a poly-time algorithm that for a given G constructs a tree decomposition of G of width at most $a_Q . \sqrt{OPT_Q(G)}$

Requirement of connectivity

- Problems monotone under removal of connected components
- For every connected component C of G

  $OPT_Q(G - C) \leq OPT_Q(G)$

VC FVS DS

B idems

Bidimensionality Intuition : Planar Vertex Cover

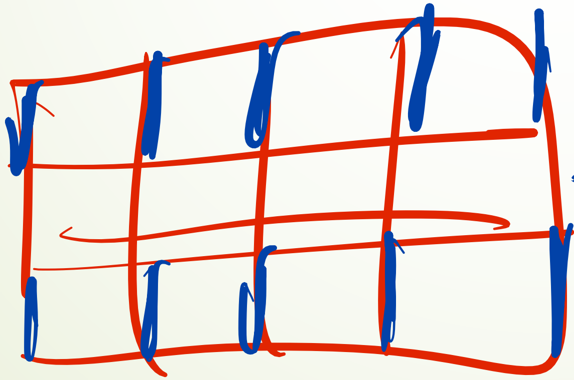
Planar G T on minors

Properties :

P1 : Lower limit on size of vertex cover of Grid(t) $\geq t^2/2$

P2 : Given a tree decomposition of width t of G, how fast can we solve Vertex Cover $2^{O(t)} n^{O(1)}$

P3 : Minor Closed Property on Vertex Cover ✓✓



$\geq t^2/2$

Bidimensionality Intuition : Planar Vertex Cover

Algorithm

$t \geq \sqrt{2k+2}$

→ find $\frac{t^2}{2} \geq k+1$ NO

For $t = \sqrt{2k+2}$ and some $\varepsilon > 0$, use of the constructive part of Planar excluded grid theorem :

- Either compute in time $O(n^2)$ a tree decomposition of G of width at most $(\frac{9}{2} + \varepsilon)t$
- Or we conclude that G has t as a minor. → NO

DP

FPT(w)

$2^{O(t)} n^{O(1)}$

→ $2^{O(\sqrt{k})} n^{O(1)}$

Bidimensionality Intuition : Planar Vertex Cover

Generalization to other problems

LP LC APM k

properties which were essential for obtaining a sub-exponential parameterized algorithm:

Arrival c(t)

P1 : Size of any solution in t is of order $\Omega(t^2)$

P2 : Given a tree decomposition of width t, the problem can be solved in time $2^{O(t)} \cdot n^{O(1)}$

FPT & 2b(t) n O(1) → +ve

P3 : Problem is minor-monotone

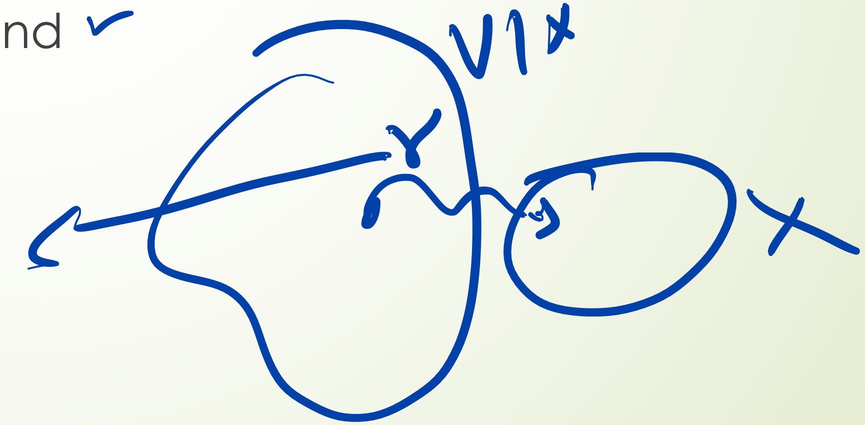
Contraction-monotone

BPM

Corollary : BP-Theorem[6]

Following parameterized problems can be solved in time $2^{O(\sqrt{k})} n^{O(1)}$:

- Planar Vertex Cover ✓ Bide mor
- Planar Independent Set ✓
- Planar Dominating Set ✓
- Planar Scattered Set for fixed d ✓ DP $2^{O(d)}$ $n^{O(1)}$
- Planar Induced Matching, and ✓
- Planar r-Center for fixed r.



Corollary : Parameter-treewidth bound[7]

+ Thm 7.10 →

$k^{O(\sqrt{k})} n^{O(1)}$ ← $t^{O(t)} n^{O(1)}$

t

Following parameterized problems can be solved in time $k^{O(\sqrt{k})}\ n^{O(1)}$:

- Planar Feedback Vertex Set
- Planar Longest Path
- Planar Longest Cycle
- Planar Cycle Packing
- Planar Connected Vertex Cover
- Planar Connected Dominating Set
- Planar Connected Feedback Vertex Set

$f(\sqrt{k})$

$2^{O(t)}$

$f(t)$

BP – Theorem[5]

Given : Bidimensional problem Q such that

Q can be solved in time $2^{O(t)} \cdot n^{O(1)}$ provided a tree decomposition of given G of width t

Then :

Q is solvable in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$ on **connected** planar graphs

Possible Extensions

- Planar excluded grid theorem can be generalized to H-minor-free graphs:
  - For every fixed graph H, t > 0, every H-minor-free graph G of treewidth more than $a_H t$ contains Grid(t) as a minor, where a_H is a constant depending on H only.
- Apex-minor-free graphs
- Minor Bidimensionality

VC FVS

$tw(G) \le \alpha_\theta \sqrt{OPT_\theta(G)}$

fixed H

$H \not\preceq G$

$\Rightarrow \boxed{tw(G) \ge \alpha_H t}$

Grid(t) minor

Apex Graph $\le G$

References

[1] Theorem 7.22 *

[2] Theorem 7.23 *

[3] Theorem 7.25 *

[4] Theorem 7.28 *

[5] Theorem 7.29 *

[6] Corollary 7.30 *

[7] Corollary 7.31 *

From Book on Parameterized Algorithms by Marek Cygan et al