Bidimensionality

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Preliminaries

- Win\Win Approaches
- Vertex-subset problem and Edge-subset problem
- φ-Minimization and Maximization Problems
- OPT
- Contraction Closed and Minor closed
- Grid(t)
- Γ(t)
- Excluded grid theorem[1]
- Planar excluded grid theorem[2]
- Planar excluded grid theorem for edge contractions[3]

Preliminaries

Win\Win Approaches

Construct a good tree decomposition of the input graph

- Small Treewidth : Dynamic programming
- Large Treewidth : Existence of an obstacle
  - No-Instance
  - Yes-Instance
  - Reduction
- Treewidth is minor-closed

Vertex-subset problem and Edge-subset problem*

Input : Graph G and a parameter k

Goal : Decide whether there exists S and $\varphi(G, S)$ = true where :

- Vertex-subset problem

  $S \subseteq V(G)$ and $|S| \leq k$

- Edge-subset problem

  $S \subseteq E(G)$ and $|S| \leq k$

* for minimization

φ-Minimization and Maximization Problems*

Input : Graph G and a parameter k

Goal : Decide whether there exists a set $S \subseteq V(G)$ and $\varphi(G, S)$ = true and :

- **φ-Minimization**

  $|S| \leq k$

- **φ-Maximization**

  $|S| \geq k$

*for vertex-subset problem

Lets define OPT

- φ-Minimization problem Q

  $OPT_Q(G) = \min\{k : (G, k) \in Q\}$

  If for no k (G, k) ∈ Q, then $OPT_Q(G)=+\infty$

- φ-Maximization problem Q

  $OPT_Q(G) = \max\{k : (G, k) \in Q\}$

  If for no k (G, k) ∈ Q, then $OPT_Q(G) = -\infty$.

Contraction Closed and Minor closed

Vertex-subset problem Q

- Contraction-closed

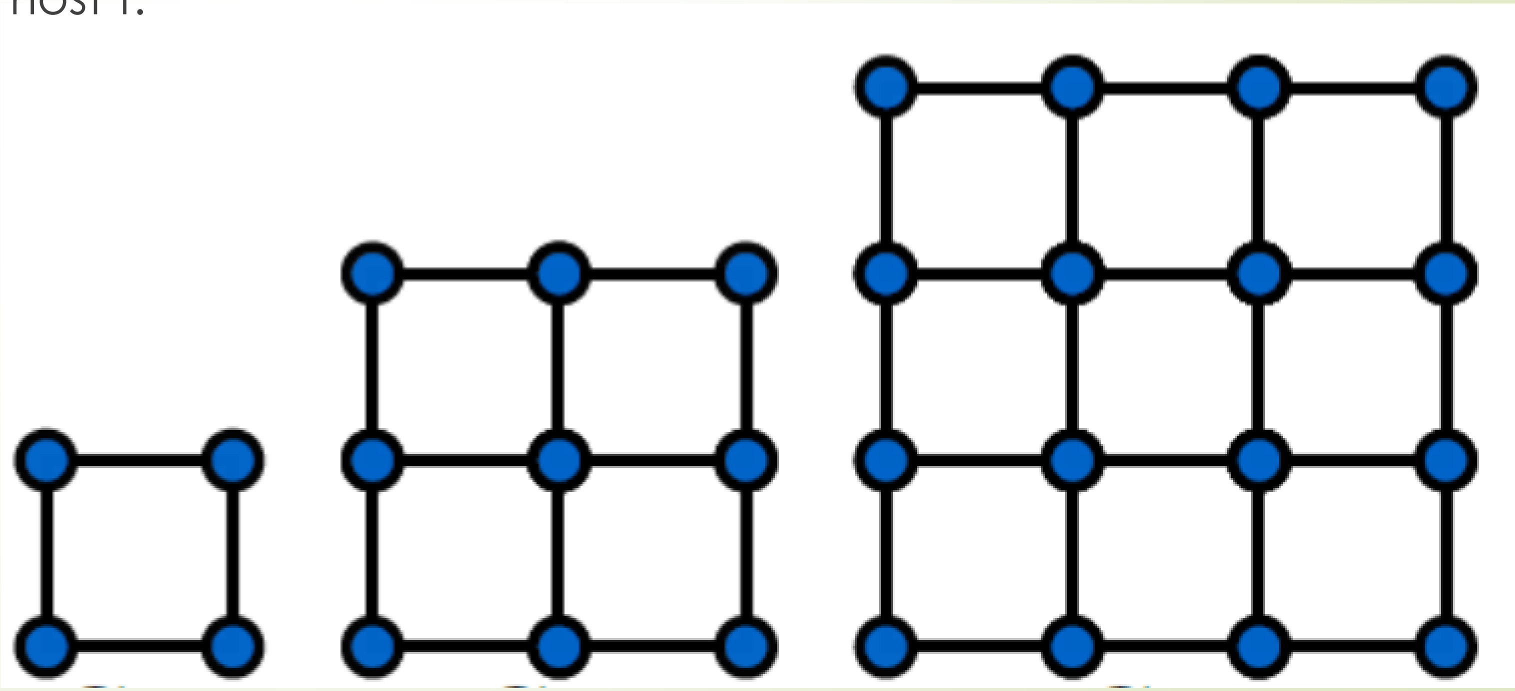
  If for every contraction H of G, $OPT_Q(H) \leq OPT_Q(G)$ for all G

- Minor-Closed

  If for every minor H of G, $OPT_Q(H) \leq OPT_Q(G)$ for all G

Grid(t)

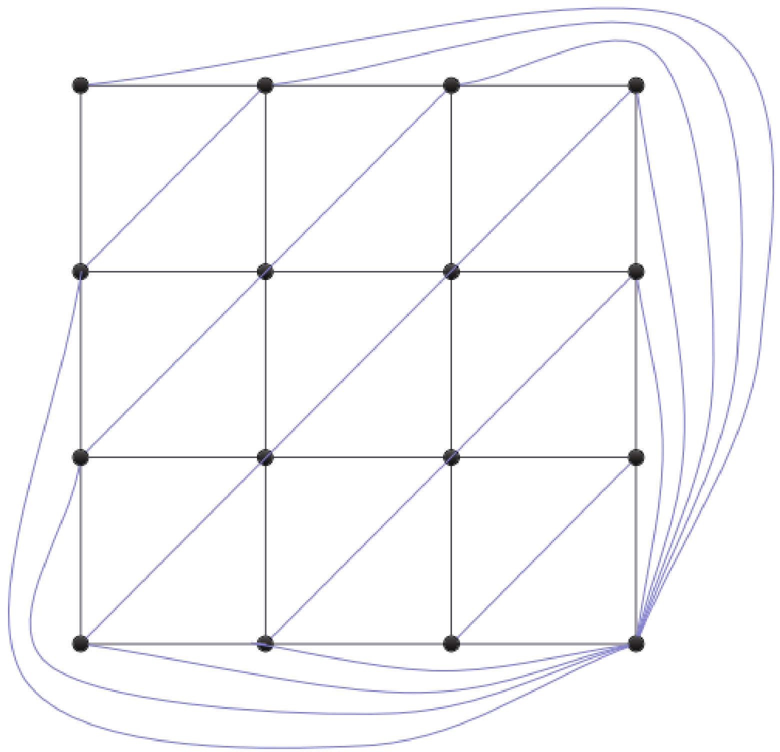
- Graph with vertex set {(x, y) : x, y ∈ {1,…,t}}. Thus t has exactly t^2 vertices.
- Two different vertices (x, y) and (x , y) are adjacent if and only if |x − x | + |y − y | = 1.
- Treewidth is at most t.



Preliminaries

Γ(t)

- Obtained from the Grid(t) by
  - adding, for all $1 \leq x, y \leq t-1$, the edge $(x+1, y),(x, y+1)$, and additionally making vertex (t, t) adjacent to all the other vertices (x, y) with $x \in \{1, t\}$ or $y \in \{1, t\}$, i.e., to the whole border of t.



Excluded grid theorem[1]

- ∃ $g(t) = O(t^{98+o(1)})$ such that every graph of treewidth larger than g(t) contains Grid(t) as minor
- Robertson and Seymour $g(t) = 2^{O(t^5)}$
- Randomized poly-time algorithm
  - Either constructs a Grid(t) minor model
  - Or finds a tree decomposition of the input graph of width at most g(t)

Lets Apply it on Cycle Packing

If G has treewidth larger than $g(t) = O(k^{49+o(1)})$:

- G contains a Grid(t) minor model for $t = 2\sqrt{k}$
- Then in this model one can find $(\sqrt{k})^2 \geq k$ vertex-disjoint cycles
- Run the approximation algorithm for parameter g(t)
  - If tw(G) > g(t), then (G, k) is a yes-instance
  - Otherwise, we obtain a tree decomposition of G of width at most 4g(t)+4

Planar excluded grid theorem[2]

- Every planar graph G of treewidth at least $9t/2$ contains Grid(t) as a minor
- For every $\varepsilon > 0$ there exists an $O(n^2)$ algorithm that, for a given n-vertex planar graph G and integer t
  - Either outputs a tree decomposition of G of width at most $(9/2 + \varepsilon)t$
  - Or constructs a minor model of Grid(t) in G

A Useful Corollary

- Treewidth of an n-vertex planar graph G is less than $\left\lceil \frac{9}{2}\sqrt{n+1} \right\rceil$
- For any ε > 0, a tree decomposition of G of width at most $\left\lceil (\frac{9}{2} + \varepsilon)\sqrt{n+1} \right\rceil$ can be constructed in $O(n^2)$ time.

Planar excluded grid theorem for edge contractions[3]

- For every **connected** planar graph G and integer $t \geq 0$, if $tw(G) \geq 9t+5$ then G contains $\Gamma(t)$ as a contraction
- For every $\varepsilon > 0$ there exists an $O(n^2)$ algorithm that, given a connected planar n-vertex graph G and integer t
  - Either outputs a tree decomposition of G of width $(9 + \varepsilon)t + 5$
  - Or a set of edges whose contraction in G results in $\Gamma(t)$

Bidimensional Problem

A vertex-subset problem Q is bidimensional if

- Q is contraction-closed
- ∃ constant $c > 0$ such that

$$OPT_Q(\Gamma(t)) \geq ct^2 \qquad \text{for every } t > 0$$

Thm : Parameter-treewidth bound[4]

Q be a bidimensional problem, then

- ∃ a constant a_Q such that for any connected planar graph G

  $tw(G) \leq a_Q \cdot \sqrt{OPT_Q(G)}$

- ∃ a poly-time algorithm that for a given G constructs a tree decomposition of G of width at most $a_Q \cdot \sqrt{OPT_Q(G)}$

Thm : Parameter-treewidth bound[4]

Proof:

BLACK BOX

Planar excluded grid theorem for edge contractions[3]

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  - Or a set of edges whose contraction in G results in $\Gamma(t)$

BP – Theorem[5]

Given : Bidimensional problem Q such that

Q can be solved in time $2^{O(t)} \cdot n^{O(1)}$

provided a tree decomposition of given G of width t

Then :

Q is solvable in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$ on **connected** planar graphs

BP – Theorem[5]

Proof:

BLACK BOX

Thm : Parameter-treewidth bound[4]

Q be a bidimensional problem, then

- ∃ a poly-time algorithm that for a given G constructs a tree decomposition of G of width at most $a_Q \cdot \sqrt{OPT_Q(G)}$

BP - Theorem

Requirement of connectivity

- Problems monotone under removal of connected components
- For every connected component C of G

  $OPT_Q(G - C) \leq OPT_Q(G)$

Bidimensionality Intuition :
Planar Vertex Cover

Properties :

P1 : Lower limit on size of vertex cover of Grid(t)

P2 : Given a tree decomposition of width t of G, how fast can we solve Vertex Cover

P3 : Minor Closed Property on Vertex Cover

Bidimensionality Intuition : Planar Vertex Cover

Algorithm

For t = $\sqrt{2k + 2}$ and some $\varepsilon > 0$, use of the constructive part of Planar excluded grid theorem :

- Either compute in time $O(n^2)$ a tree decomposition of G of width at most (9 2 + ε)t
- Or we conclude that G has t as a minor.

Generalization to other problems

properties which were essential for obtaining a sub-exponential parameterized algorithm:

P1 : Size of any solution in t is of order $\Omega(t^2)$

P2 : Given a tree decomposition of width t, the problem can be solved in time $2^{O(t)} \cdot n^{O(1)}$

P3 : Problem is minor-monotone

Corollary : BP-Theorem[6]

Following parameterized problems can be solved in time $2^{O(\sqrt{k})} n^{O(1)}$:

- Planar Vertex Cover
- Planar Independent Set
- Planar Dominating Set
- Planar Scattered Set for fixed d
- Planar Induced Matching, and
- Planar r-Center for fixed r.

Corollary : Parameter-treewidth bound[7]

Following parameterized problems can be solved in time $k^{O(\sqrt{k})} n^{O(1)}$:

- Planar Feedback Vertex Set
- Planar Longest Path
- Planar Longest Cycle
- Planar Cycle Packing
- Planar Connected Vertex Cover
- Planar Connected Dominating Set
- Planar Connected Feedback Vertex Set

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Given : Bidimensional problem Q such that

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Then :

Q is solvable in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$ on **connected** planar graphs

Possible Extensions

- Planar excluded grid theorem can be generalized to H-minor-free graphs:
  - For every fixed graph H, $t > 0$, every H-minor-free graph G of treewidth more than $a_H t$ contains Grid(t) as a minor, where a_H is a constant depending on H only.
- Apex-minor-free graphs
- Minor Bidimensionality

References

[1] Theorem 7.22 *

[2] Theorem 7.23 *

[3] Theorem 7.25 *

[4] Theorem 7.28 *

[5] Theorem 7.29 *

[6] Corollary 7.30 *

[7] Corollary 7.31 *

From Book on Parameterized Algorithms by Marek Cygan et al