On Guillotine Separable Packings for the Two-dimensional Geometric Knapsack Problem

Arnab Maiti, Indian Institute of Technology, Kharagpur

Joint work with

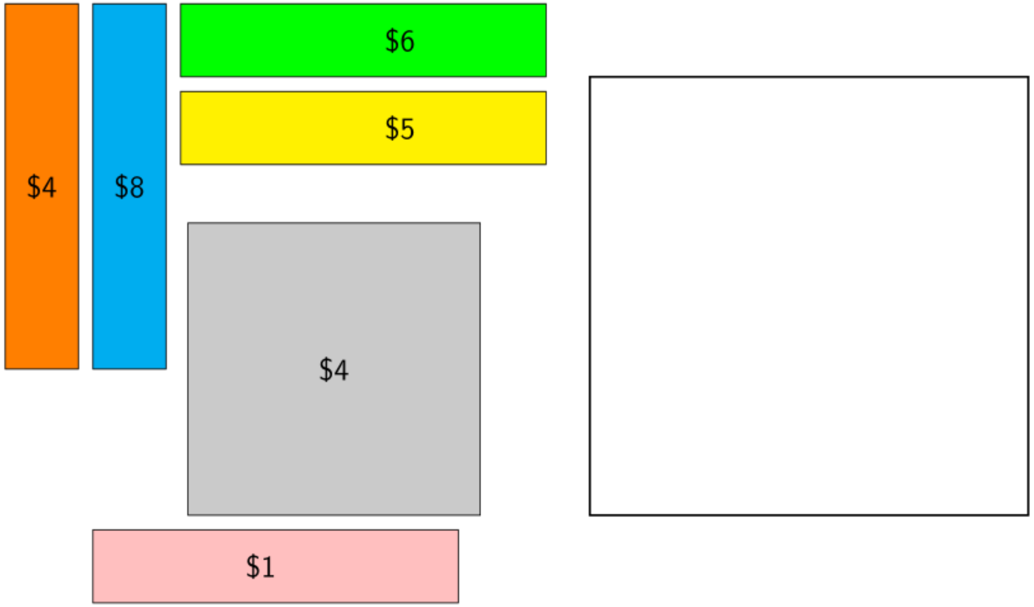
Arindam Khan, Indian Institute of Science, Bangalore

Amatya Sharma, Indian Institute of Technology, Kharagpur

Andreas Wiese, Universidad de Chile

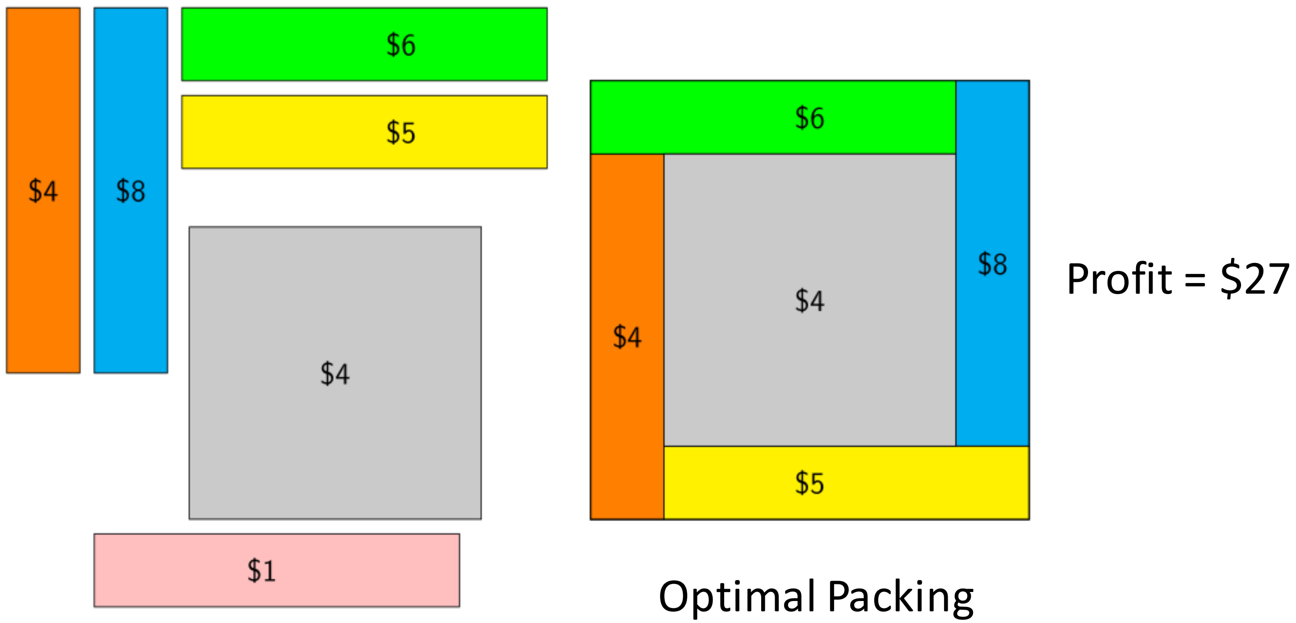
2D Geometric Knapsack

- Input:
  - Knapsack: $N \times N$ Square where N is an integer.
  - Items: Axis parallel rectangles with associated integral height, width and profit.
- Goal: Pack most profitable non-overlapping subset of items.



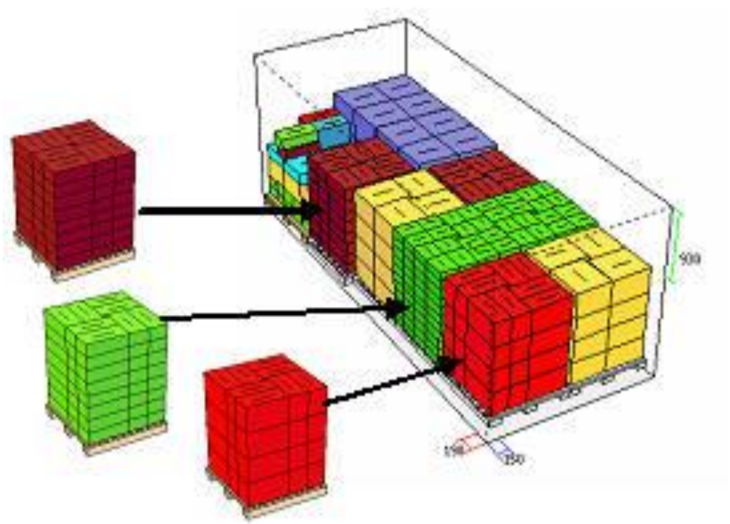
2D Geometric Knapsack

- Input:
  - Knapsack: $N \times N$ Square where N is an integer.
  - Items: Axis parallel rectangles with associated integral height, width and profit.
- Goal: Pack most profitable non-overlapping subset of items.

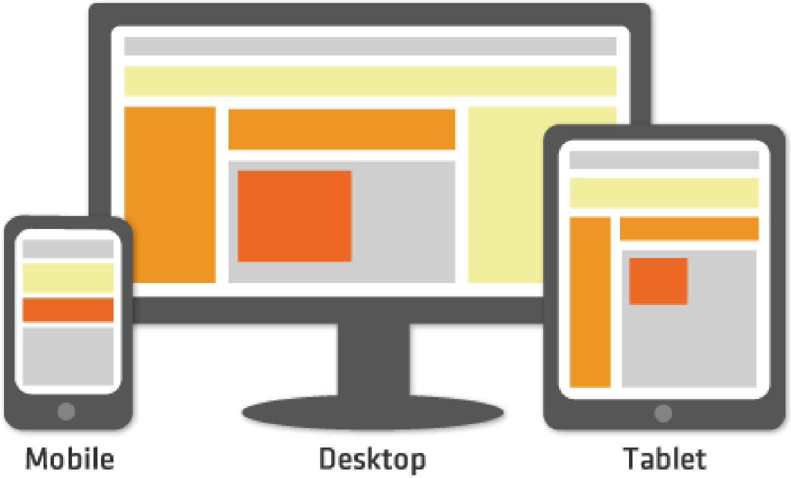


Optimal Packing

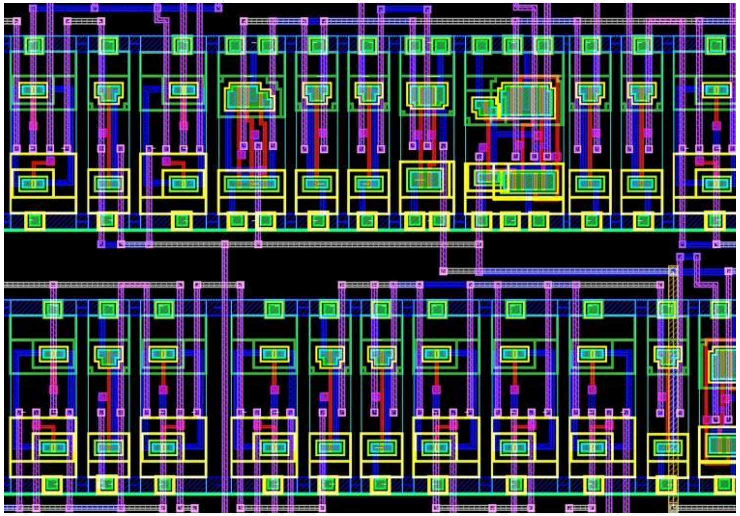
Application of 2D Geometric Knapsack



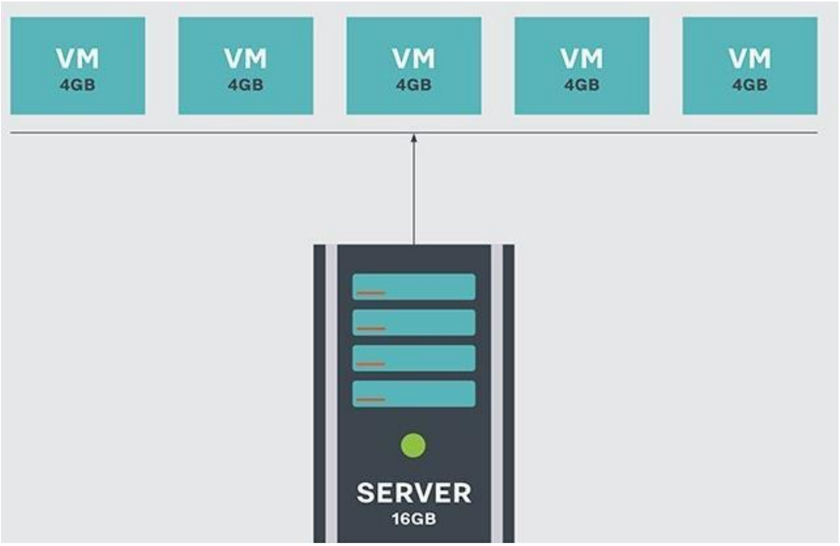
Logistics: Optimal Truck Loading



Advertisement Placement



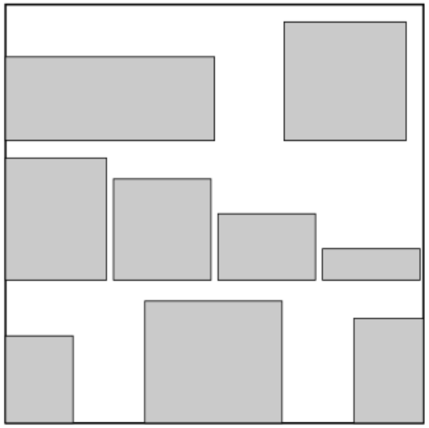
VLSI Design



Memory Allocation

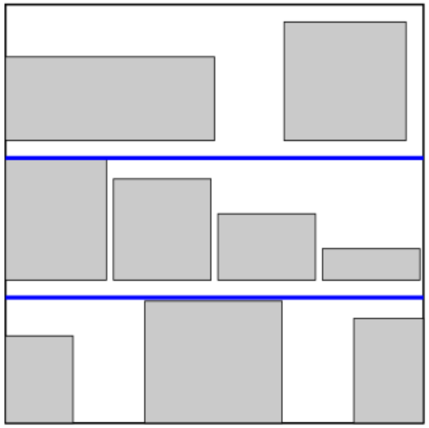
Guillotine Separability

A packing is said to be Guillotine separable if each item can be cut out using axis parallel end-to-end cuts (also known as guillotine cuts).



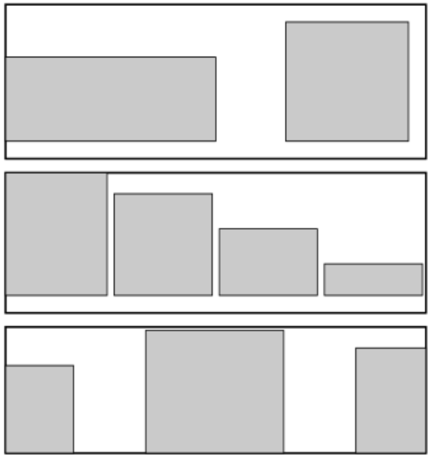
Guillotine Separability

A packing is said to be Guillotine separable if each item can be cut out using axis parallel end-to-end cuts (also known as guillotine cuts).



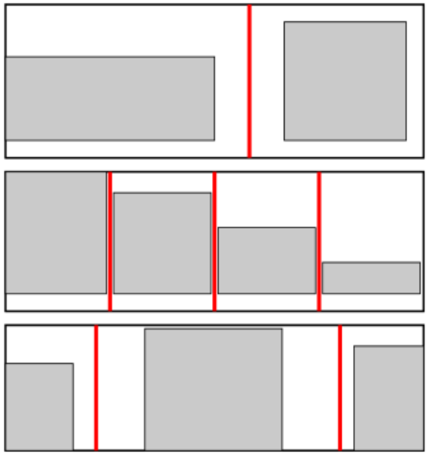
Guillotine Separability

A packing is said to be Guillotine separable if each item can be cut out using axis parallel end-to-end cuts (also known as guillotine cuts).



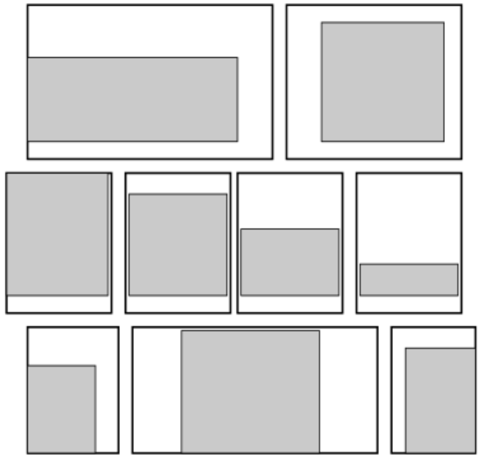
Guillotine Separability

A packing is said to be Guillotine separable if each item can be cut out using axis parallel end-to-end cuts (also known as guillotine cuts).

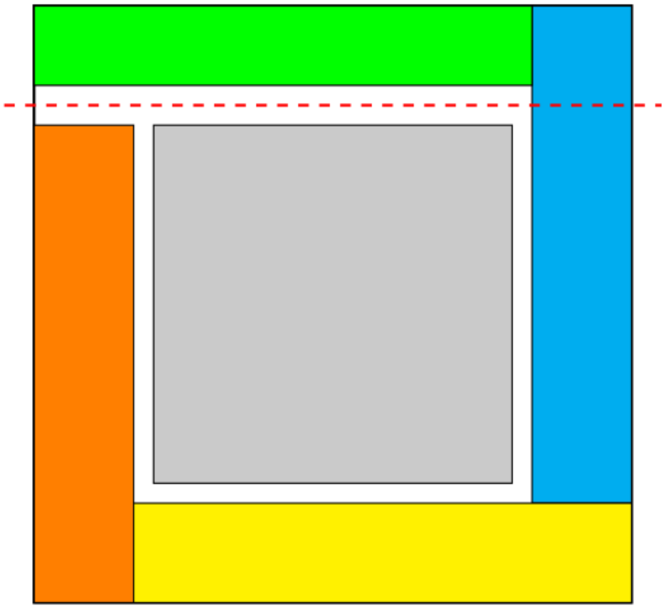


Guillotine Separability

A packing is said to be Guillotine separable if each item can be cut out using axis parallel end-to-end cuts (also known as guillotine cuts).



Non-guillotine separable packing

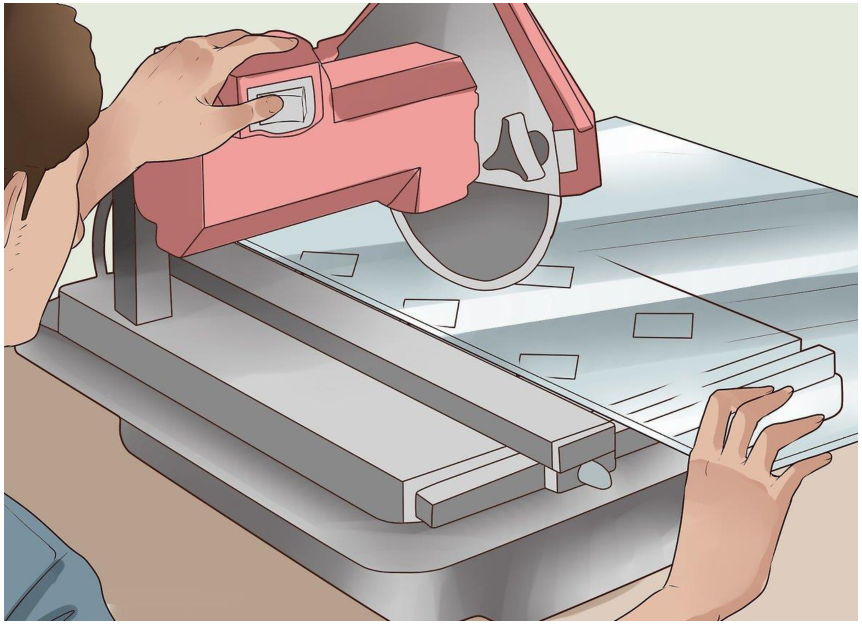


The packing is not a guillotine packing as any end-to-end cut in the knapsack intersects at least one of the packed rectangles.

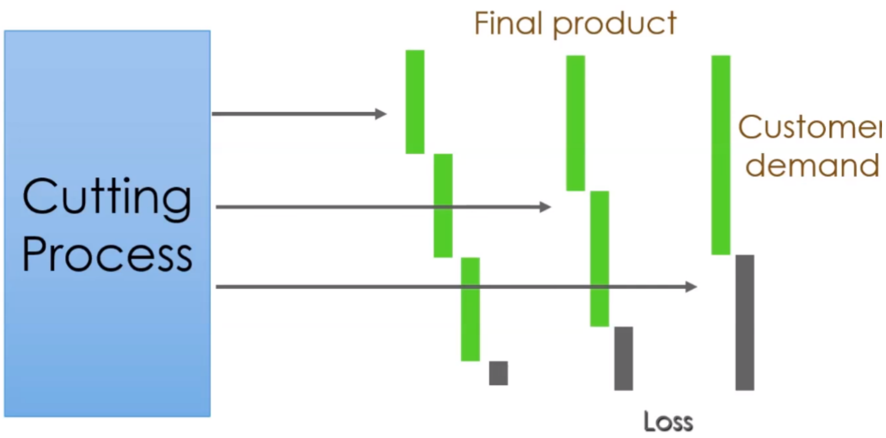
Application of Guillotine Cuts

Paper Cutting Industry

Crepe-Rubber Industry



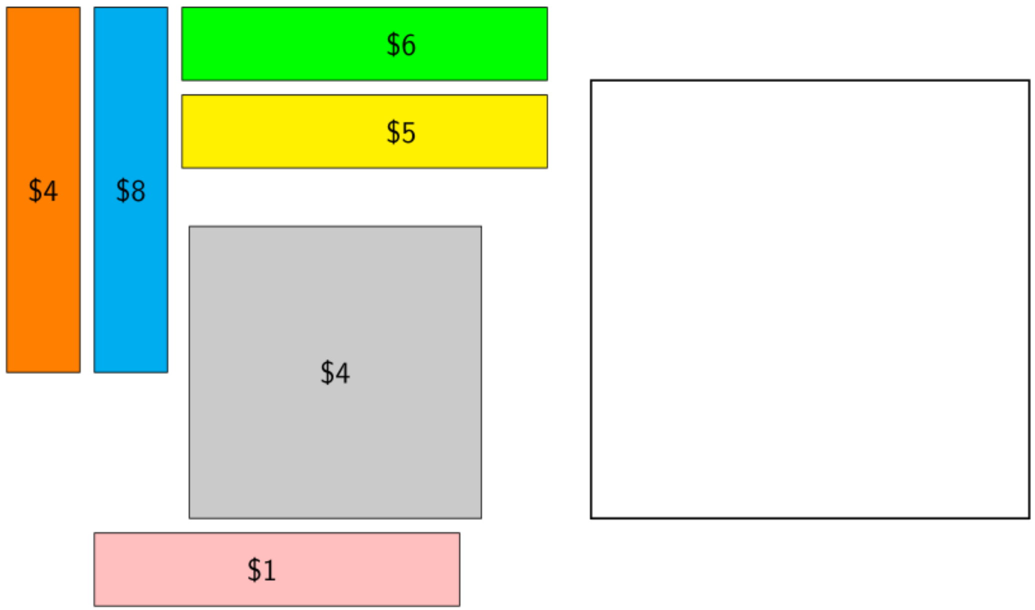
Glass Cutting Industry



Using Guillotine cuts reduces the cost and simplifies the process

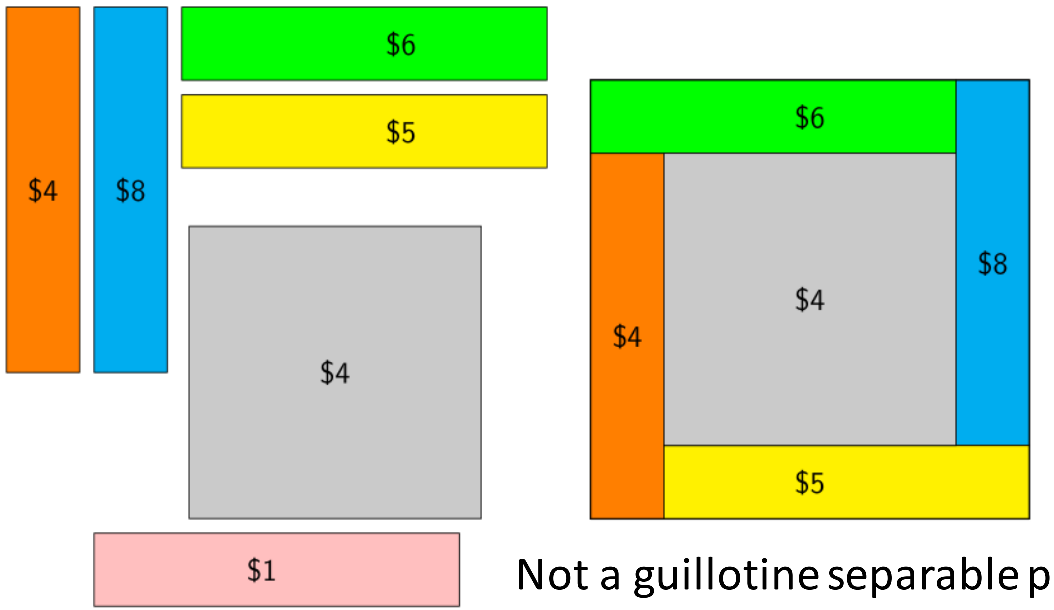
2D Guillotine Knapsack

- Input:
  - Knapsack: $N \times N$ Square where N is an integer.
  - Items: Axis parallel rectangles with associated integral height, width and profit.
- Goal: Pack most profitable subset of items s.t. items are
  - Non-Overlapping
  - Guillotine Separable



2D Guillotine Knapsack

- Input:
  - Knapsack: $N \times N$ Square where N is an integer.
  - Items: Axis parallel rectangles with associated integral height, width and profit.
- Goal: Pack most profitable subset of items s.t. items are
  - Non-Overlapping
  - Guillotine Separable

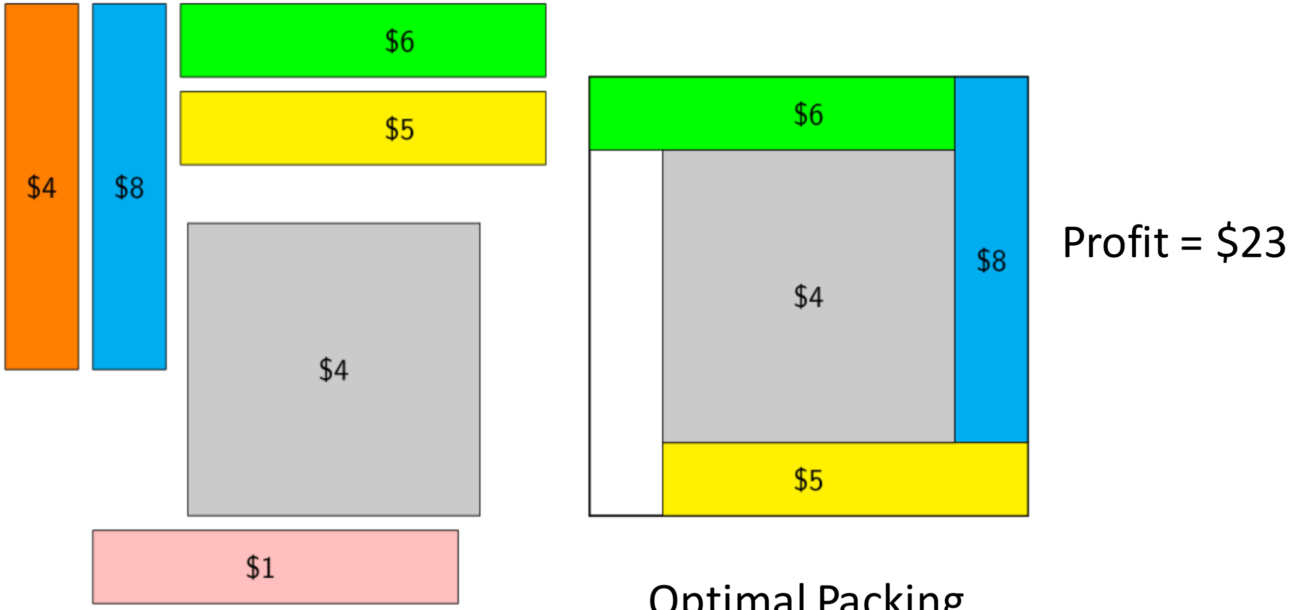


Profit = $27

Not a guillotine separable packing

2D Guillotine Knapsack

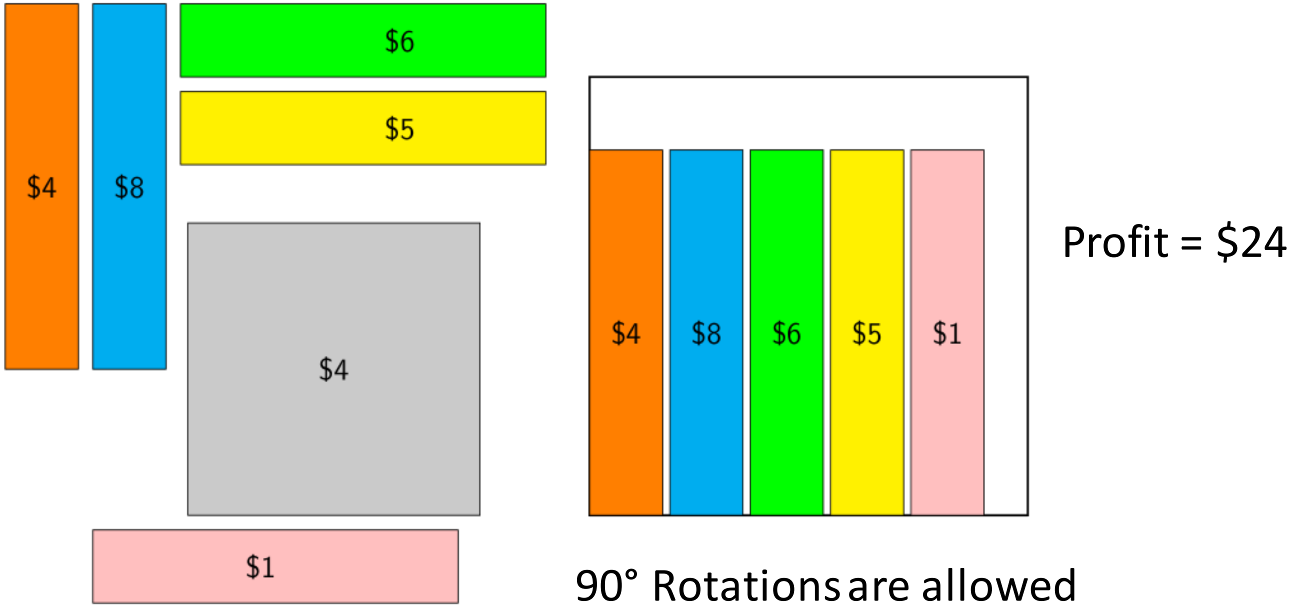
- Input:
  - Knapsack: $N \times N$ Square where N is an integer.
  - Items: Axis parallel rectangles with associated integral height, width and profit.
- Goal: Pack most profitable subset of items s.t. items are
  - Non-Overlapping
  - Guillotine Separable



Optimal Packing

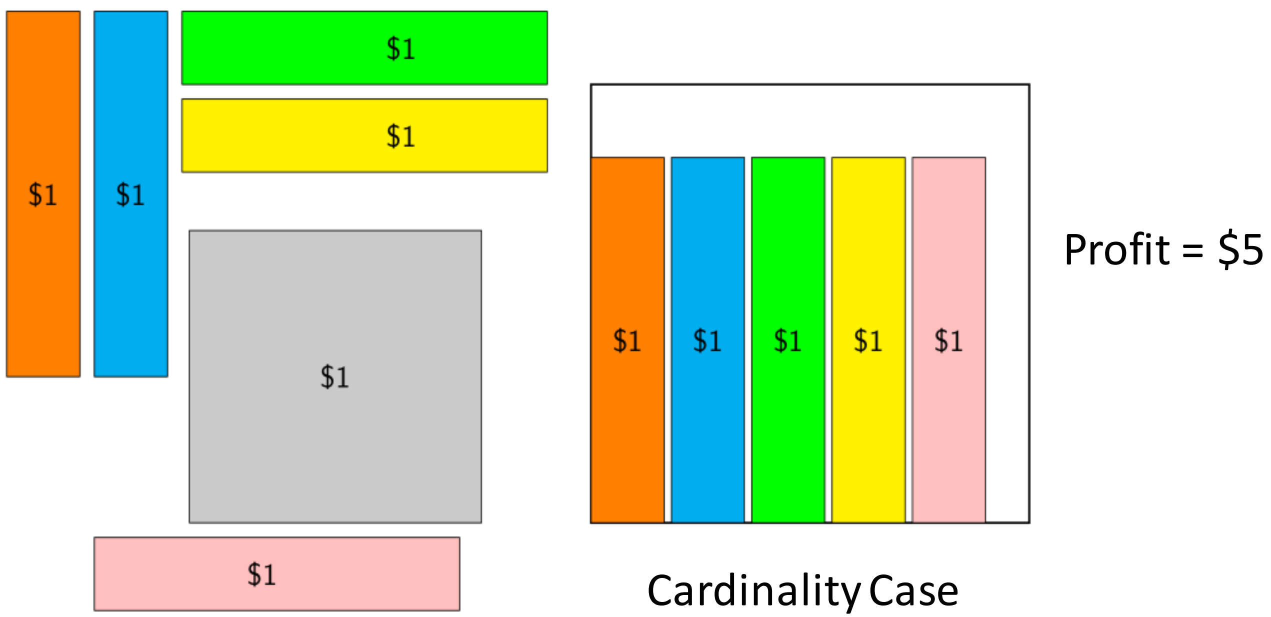
2D Guillotine Knapsack

- Input:
  - Knapsack: $N \times N$ Square where N is an integer.
  - Items: Axis parallel rectangles with associated integral height, width and profit.
- Goal: Pack most profitable subset of items s.t. items are
  - Non-Overlapping
  - Guillotine Separable



2D Guillotine Knapsack

- Input:
  - Knapsack: $N \times N$ Square where N is an integer.
  - Items: Axis parallel rectangles with associated integral height, width and profit.
- Goal: Pack most profitable subset of items s.t. items are
  - Non-Overlapping
  - Guillotine Separable



Other related problems

- Both 2D Bin Packing (Bansal et al. FOCS'05) and 2D Strip Packing (Seiden et al. Mathematical Programing'05) are well-studied in the guillotine setting.
- Pach-Tardos conjecture: For any set of n non-overlapping axis-parallel rectangles, there is a guillotine cutting sequence separating $\Omega(n)$ of them.
- Maximum Independent Set of Rectangles (MISR) problem

Prior Results for 2D Guillotine Knapsack

- NP-Hard as it is a generalized version of 1D Knapsack Problem
- $(3 + \varepsilon)$-approximation algorithm [Jansen and Zhang, SODA'04]
- Cardinality Case: QPTAS with quasi-polynomially bounded input, i.e. $N = n^{O(\log n)}$ [Abed et al, APPROX'15]

Our Results for 2D Guillotine Knapsack

- Pseudo-Polynomial Time Approximation Scheme (PPTAS) for all the variants, i.e, $(1+\varepsilon)$-approximation with pseudo-polynomial running time of $(nN)^{O_\varepsilon(1)}$.
- Note that if the size of the knapsack N is polynomially bounded in the number of items n, i.e. $N = n^{O(1)}$ then we have a Polynomial Time Approximation Scheme (PTAS).

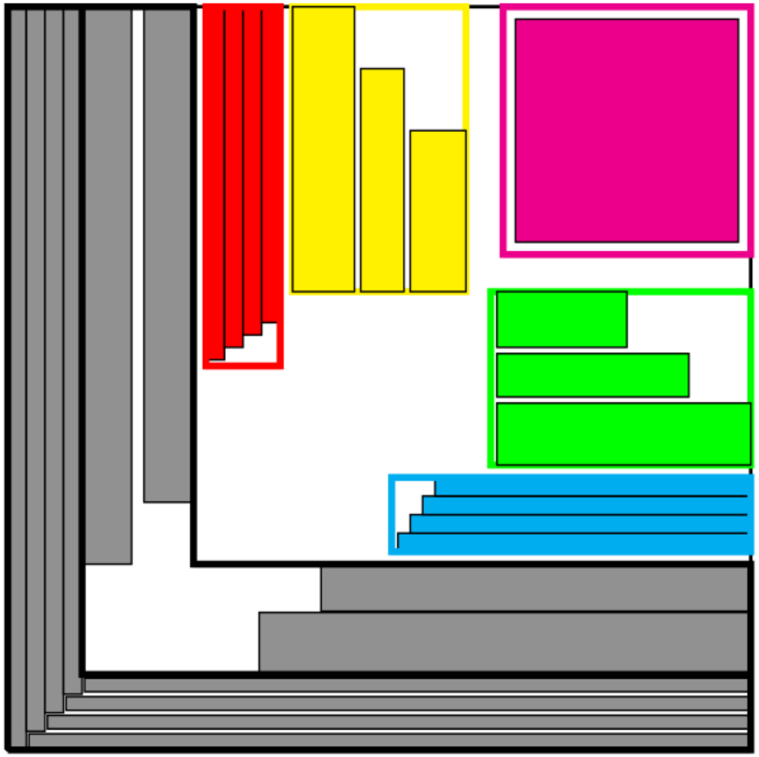
Our Techniques

- Structural Lemma: Existence of near-optimal nicely structured solutions
- Guessing the Packing: Guess the nice packing in $(nN)^{O_\varepsilon(1)}$ time

Structural Lemma

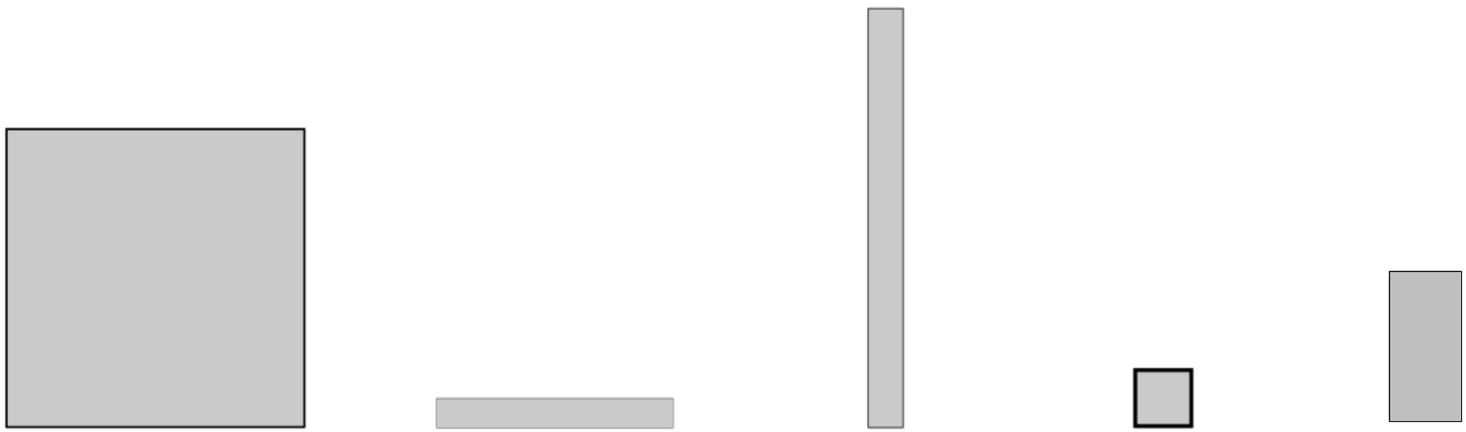
Structural Lemma

There exists a near optimal nice packing of horizontal, vertical and large rectangles into $O_\varepsilon(1)$ pseudo guillotine separable compartments.



Classification of Rectangular items

Using the constants $0 < \mu < \delta \le \varepsilon$, we classify each item i with width w_i and height h_i as follows.



Large:	Horizontal:	Vertical:	Small:	Intermediate:
$w_i > \delta N$	$w_i > \delta N$	$w_i \le \mu N$	$w_i \le \mu N$	Others
$h_i > \delta N$	$h_i \le \mu N$	$h_i > \delta N$	$h_i \le \mu N$	

Classification of Rectangular items

Using the constants $0 < \mu < \delta \leq \varepsilon$, we classify each item i with width w_i and height h_i as follows.

Large:
$w_i > \delta N$
$h_i > \delta N$

Horizontal:
$w_i > \delta N$
$h_i \leq \mu N$

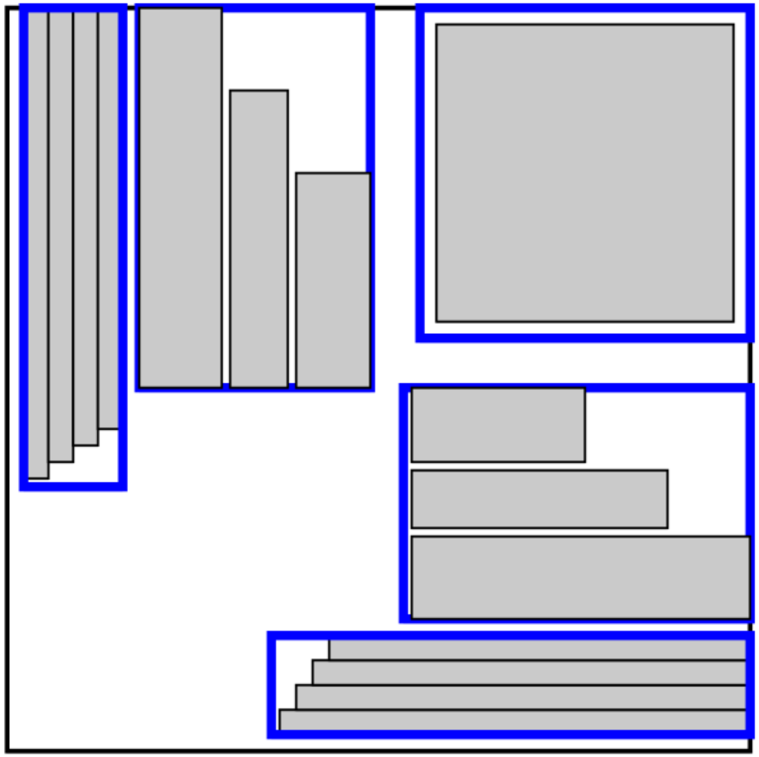
Vertical:
$w_i \leq \mu N$
$h_i > \delta N$

Small:
$w_i \leq \mu N$
$h_i \leq \mu N$

Intermediate:
Others
(Negligible
Profit)

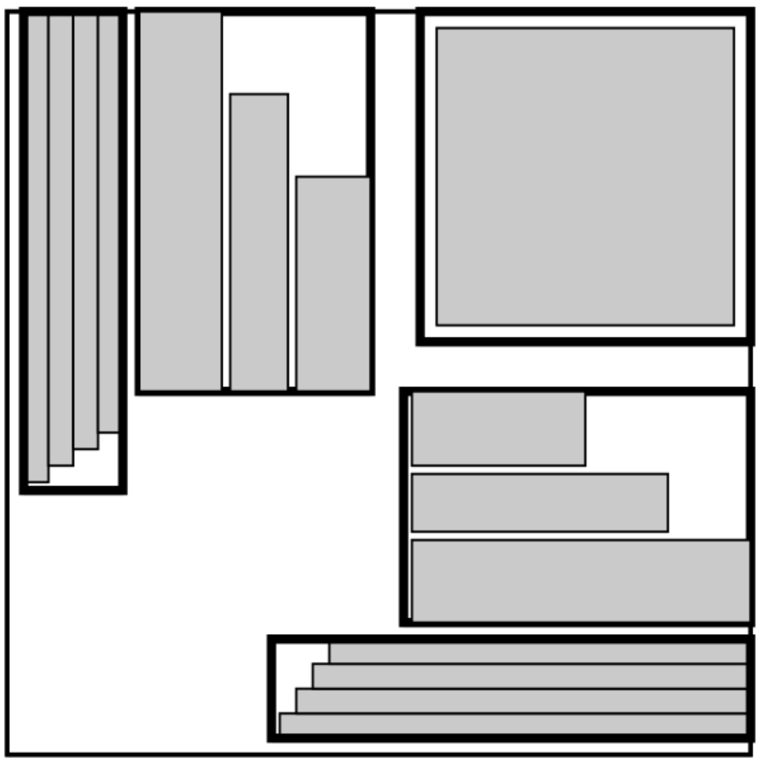
Nicely packed compartments

We first define some special regions called as compartments which is quite integral to our structural results.
First kind of region is a rectangular region called box compartment.



Nicely packed box compartments

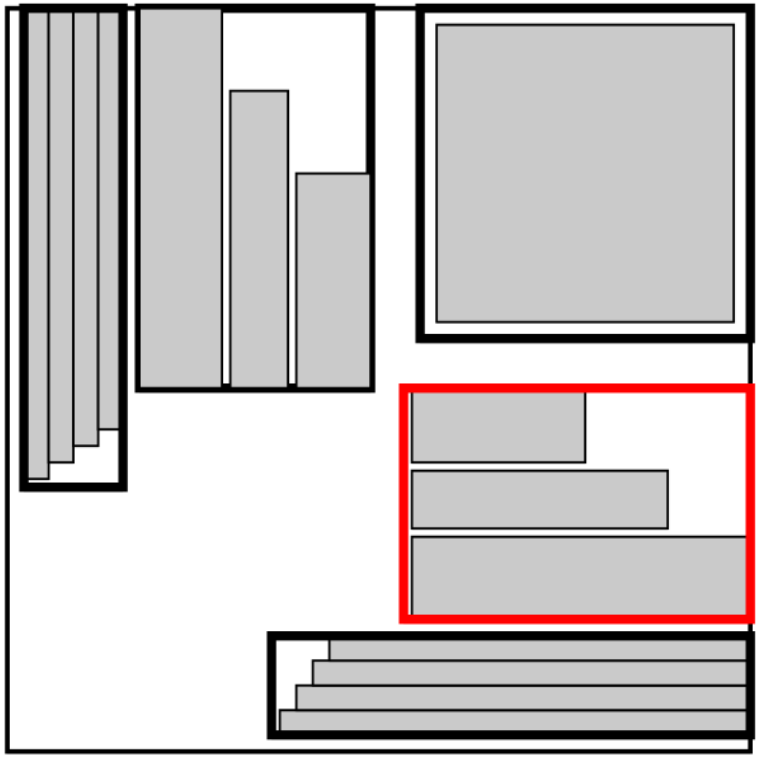
Items in a box compartment B is said to be nicely packed if one of the following happens.



Nicely packed box compartments

Items in a box compartment B is said to be nicely packed if one of the following happens.

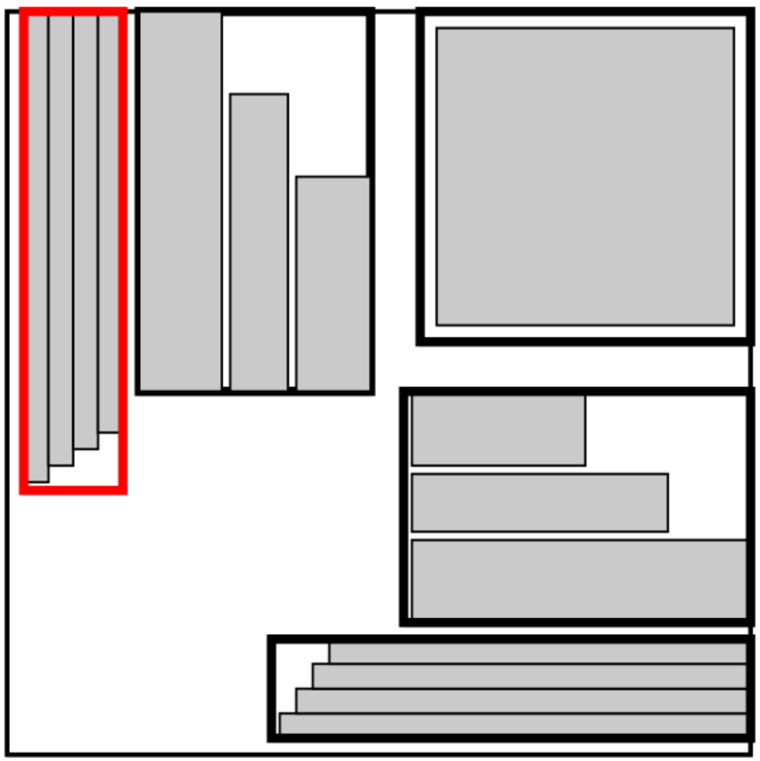
- Either all the items inside B are horizontal and the items are placed on top of each other



Nicely packed box compartments

Items in a box compartment B is said to be nicely packed if one of the following happens.

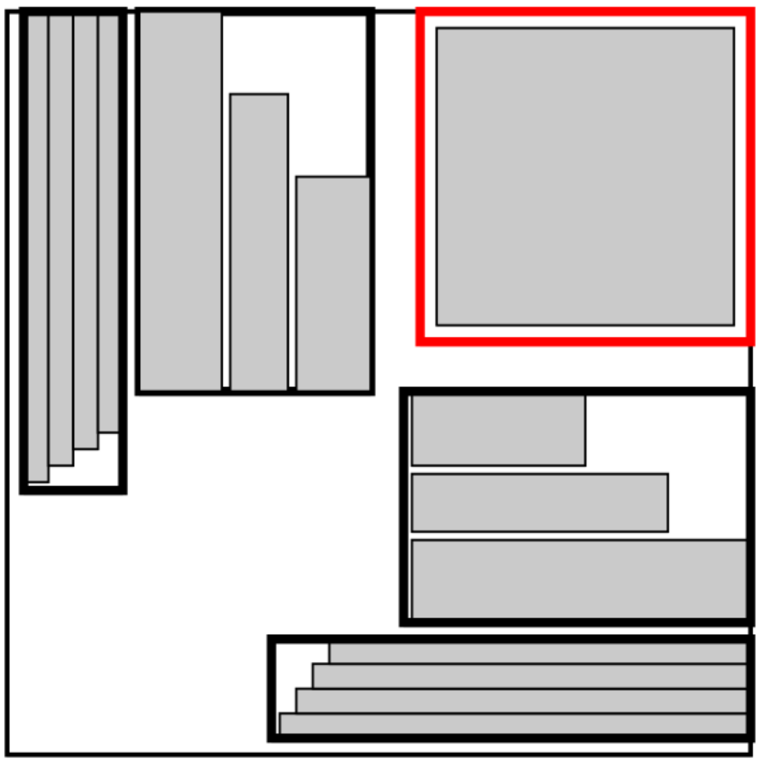
- Either all the items inside B are horizontal and the items are placed on top of each other
- Or all the items inside B are vertical and the items are placed side by side



Nicely packed box compartments

Items in a box compartment B is said to be nicely packed if one of the following happens.

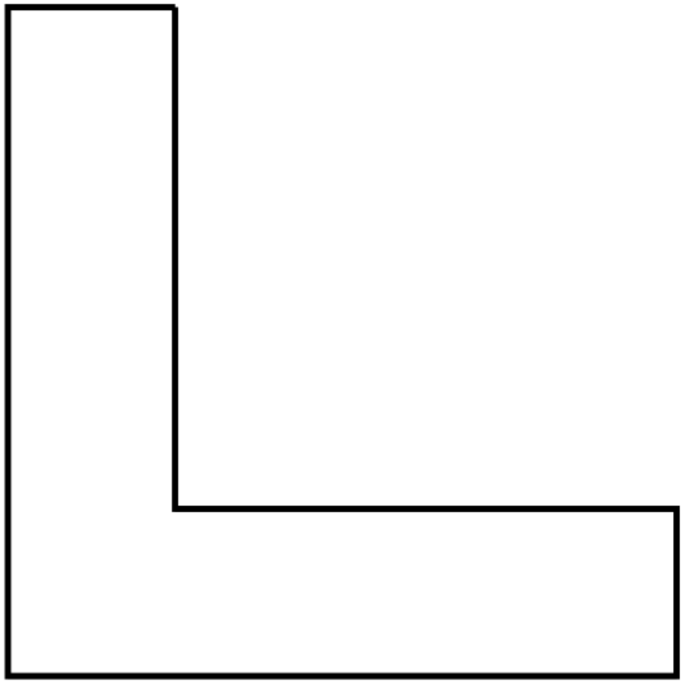
- Either all the items inside B are horizontal and the items are placed on top of each other
- Or all the items inside B are vertical and the items are placed side by side
- Or B contains only one large item.



Nicely packed **L**-compartment

Second kind of region is the region in the shape of **L** called **L**-compartment.

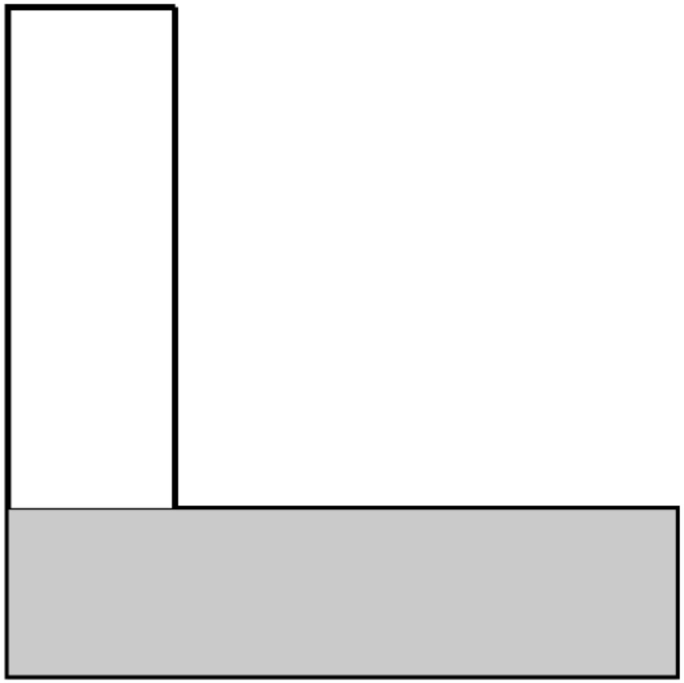
An **L**-compartment has two parts:



Nicely packed **L**-compartment

Second kind of region is the region in the shape of **L** called **L**-compartment.
An **L**-compartment has two parts:

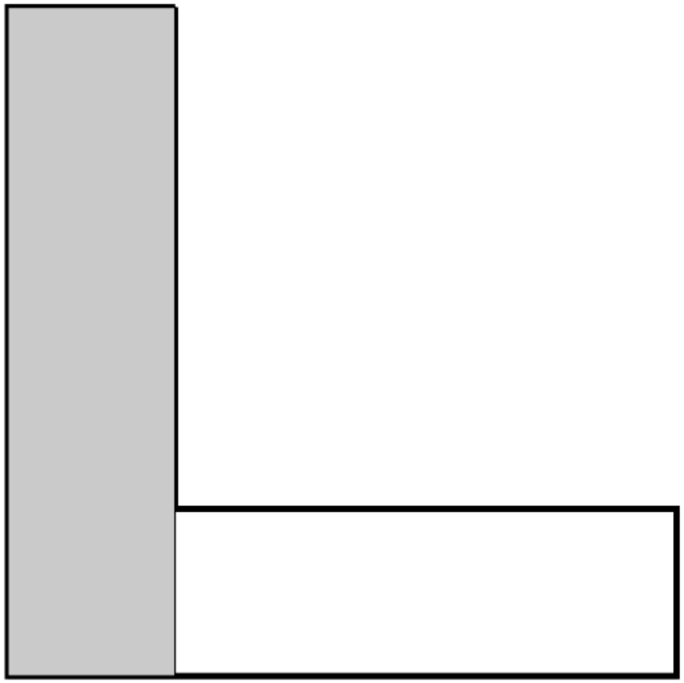
- Horizontal Part



Nicely packed **L**-compartment

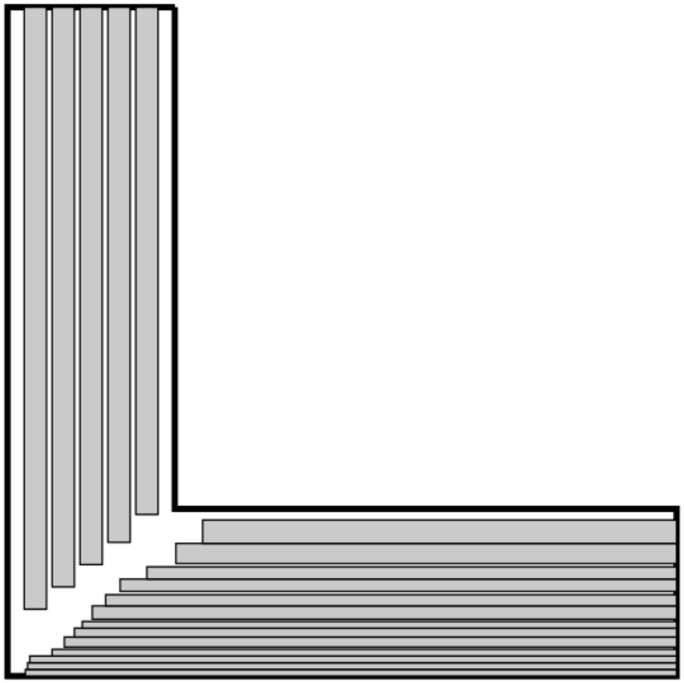
Second kind of region is the region in the shape of **L** called **L**-compartment.
An **L**-compartment has two parts:

- Horizontal Part
- Vertical Part



Nicely packed **L**-compartment

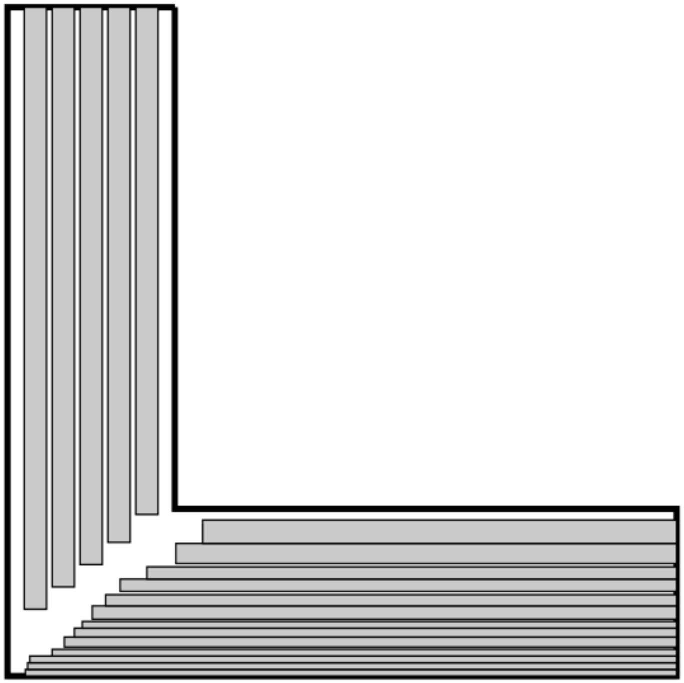
Items in the **L**-compartment is said to be nicely packed if all the following conditions hold true



Nicely packed **L**-compartment

Items in the **L**-compartment is said to be nicely packed if all the following conditions hold true

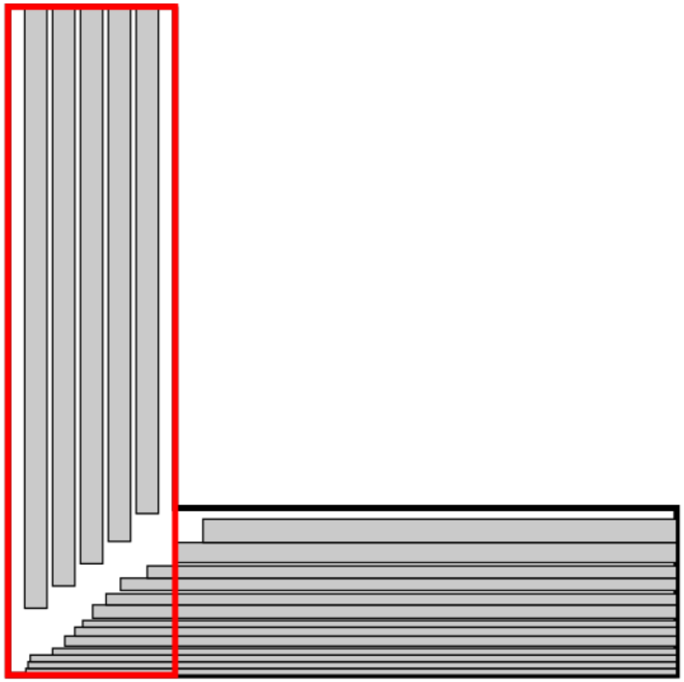
- Items don't overlap with each other.



Nicely packed **L**-compartment

Items in the **L**-compartment is said to be nicely packed if all the following conditions hold true

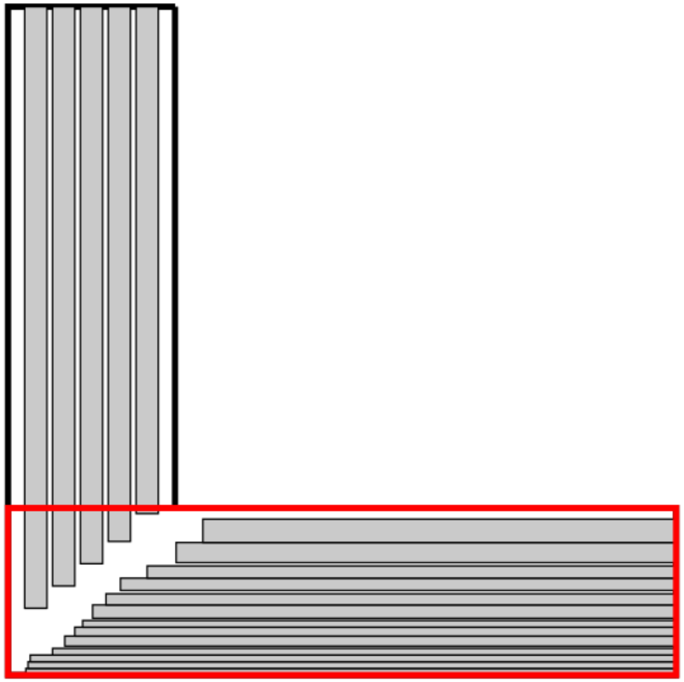
- Items don't overlap with each other.
- Vertical Items in the vertical part are nicely packed.



Nicely packed **L**-compartment

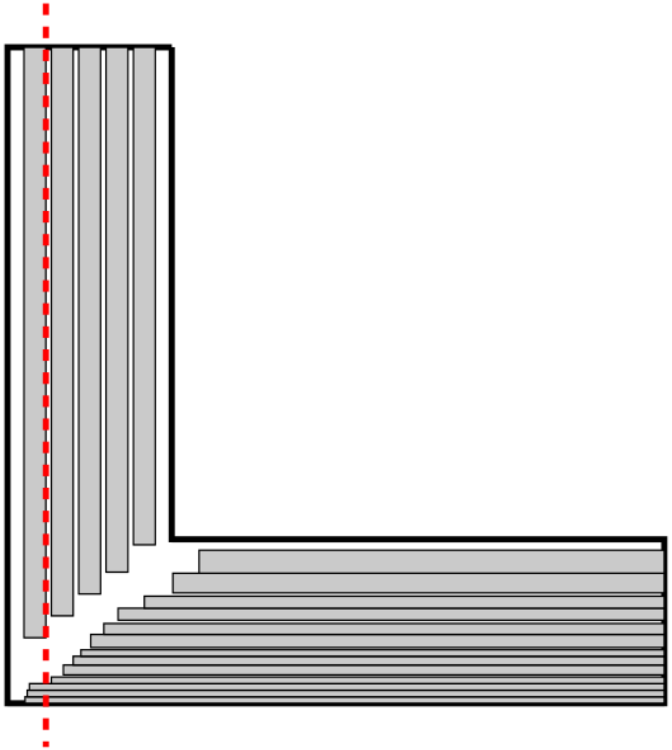
Items in the **L**-compartment is said to be nicely packed if all the following conditions hold true

- Items don't overlap with each other.
- Vertical Items in the vertical part are nicely packed.
- Horizontal Items in the horizontal part are also nicely packed.



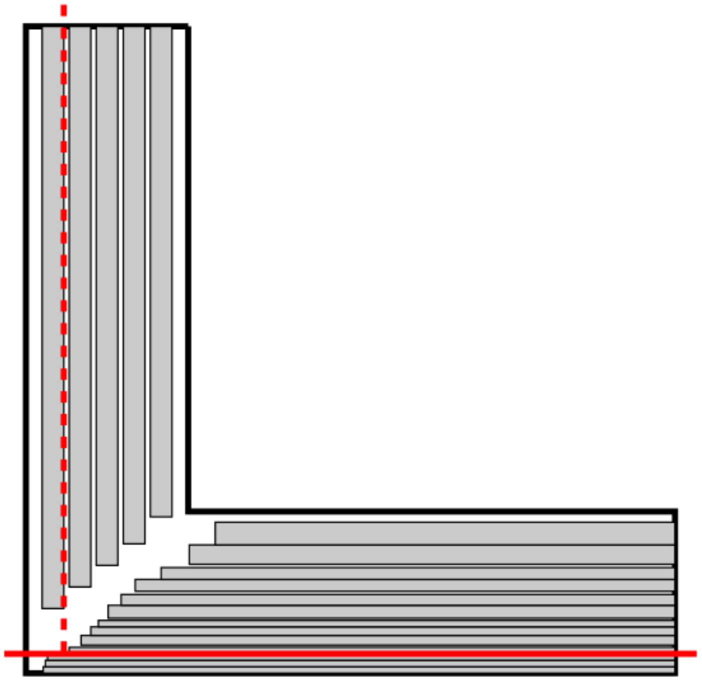
Nicely Packed **L**-compartment is guillotine separable

Consider the left most vertical cut. Note that it intersects some horizontal rectangles I'_{hor}.



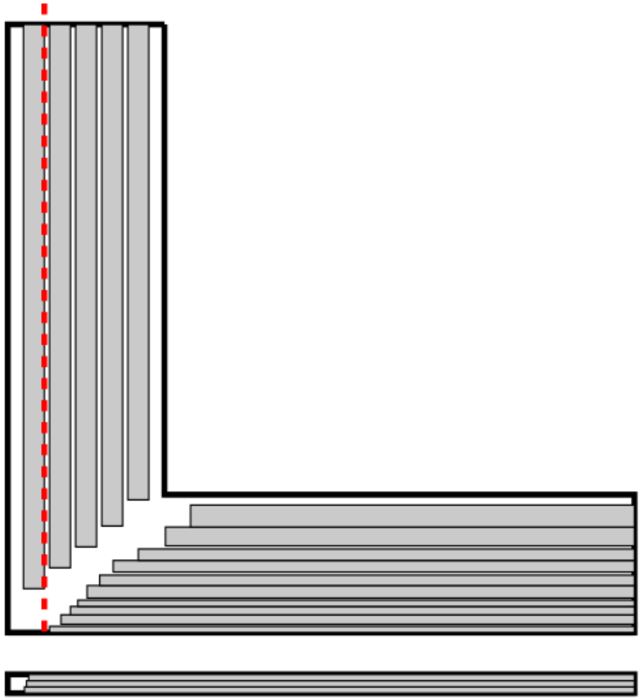
Nicely Packed **L**-compartment is guillotine separable

However, the horizontal red cut separates the items in I'_{hor} from the other horizontal items without intersecting any vertical item.



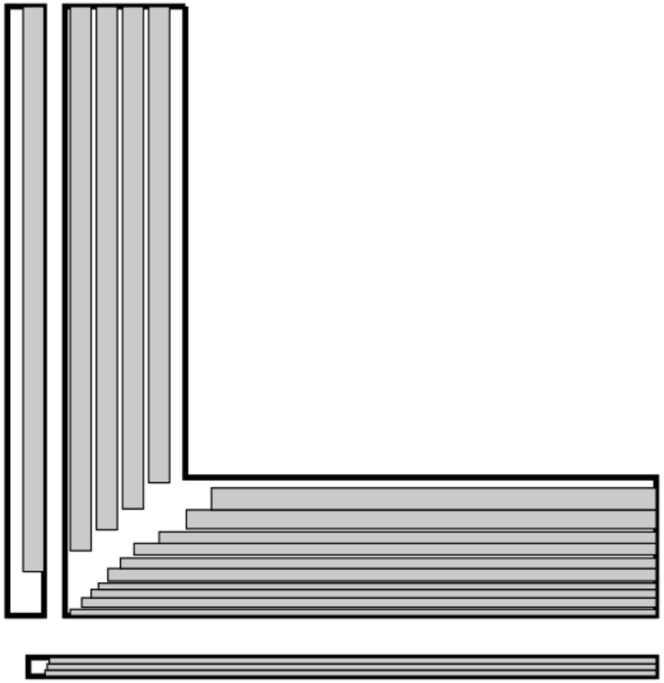
Nicely Packed **L**-compartment is guillotine separable

Now the vertical cut is a valid guillotine cut.

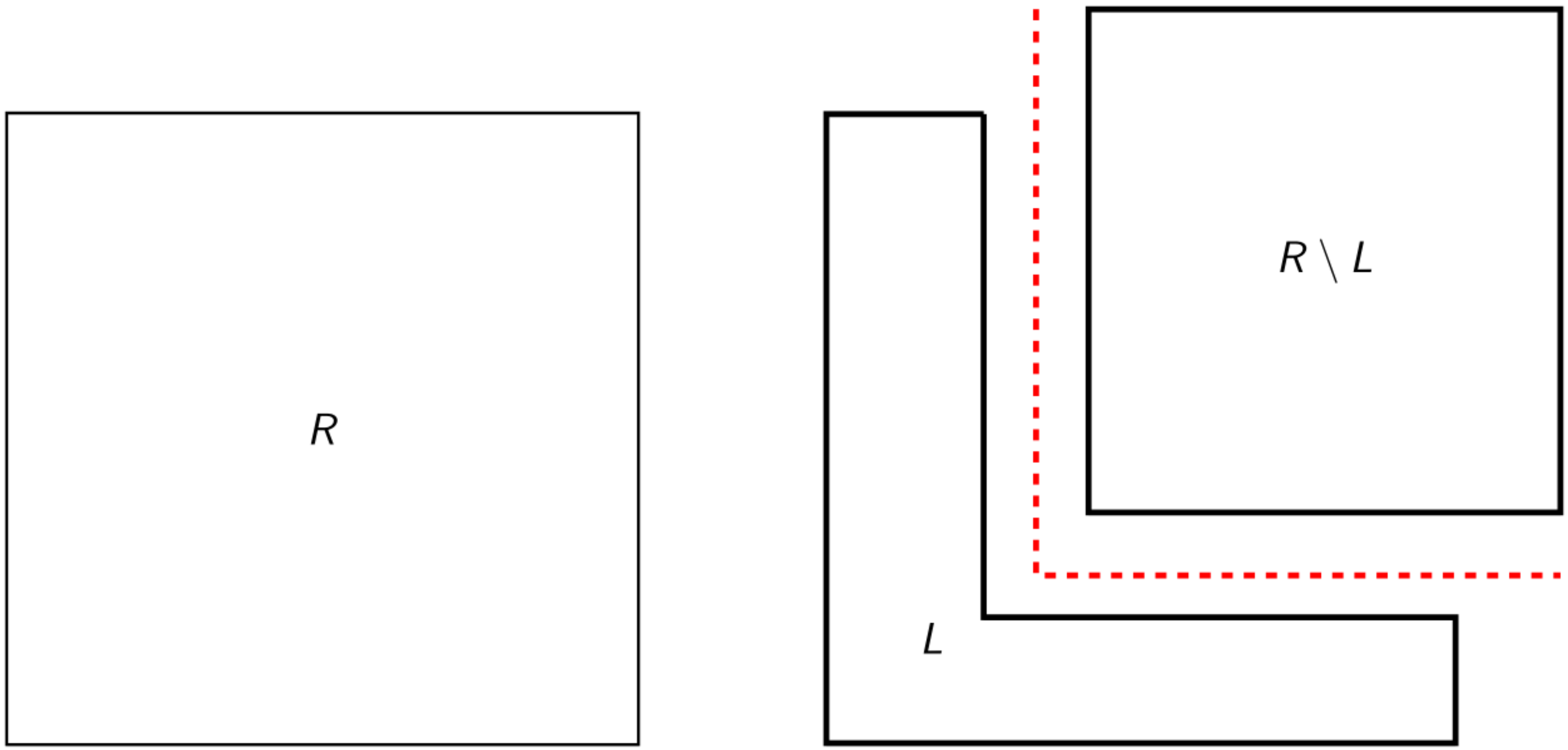


Nicely Packed **L**-compartment is guillotine separable

Repeating the process for the smaller **L**-compartment will separate out all the rectangles.

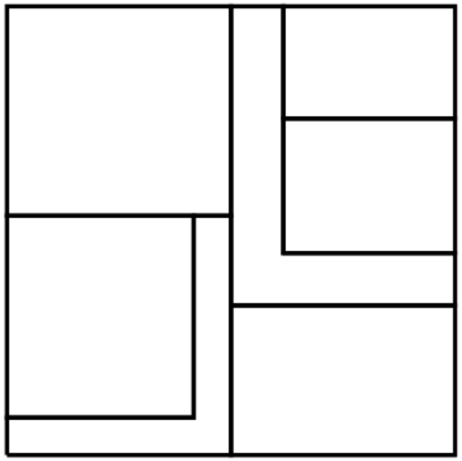


Pseudo Guillotine Cuts



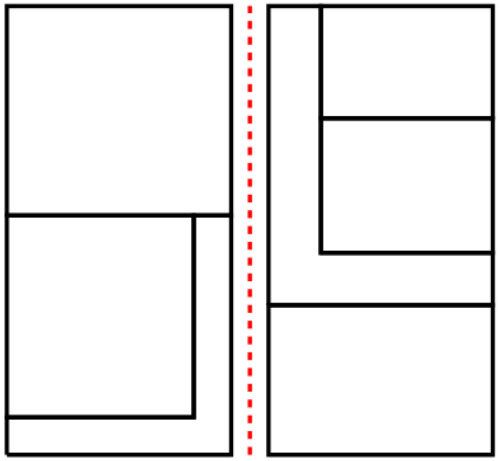
We divide R into L and $R \setminus L$ by using a cut of the shape L. We denote this cut as pseudo guillotine cut.

Pseudo Guillotine Separability



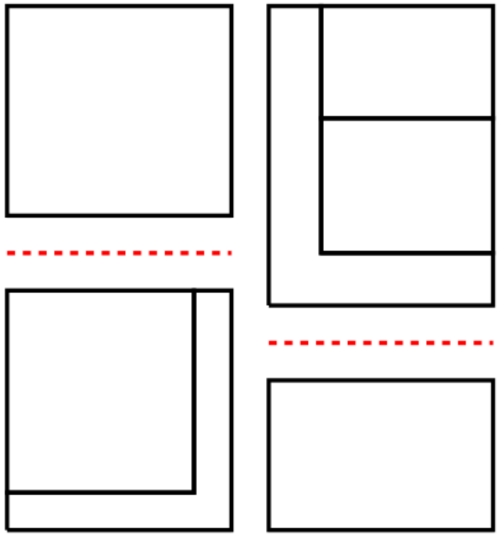
A group of compartments is said to be pseudo guillotine separable if all the compartments can be extracted using pseudo guillotine cuts and guillotine cuts.

Pseudo Guillotine Separability



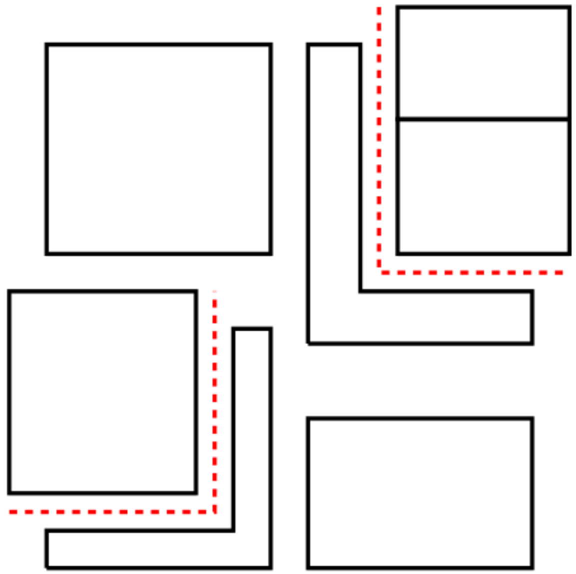
A group of compartments is said to be pseudo guillotine separable if all the compartments can be extracted using pseudo guillotine cuts and guillotine cuts.

Pseudo Guillotine Separability



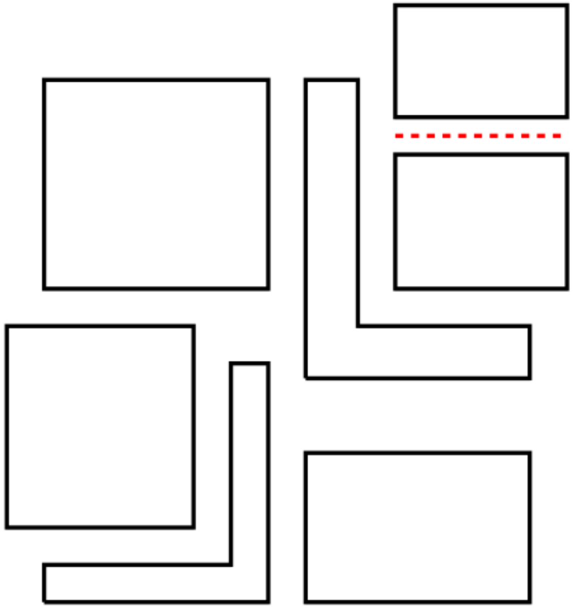
A group of compartments is said to be pseudo guillotine separable if all the compartments can be extracted using pseudo guillotine cuts and guillotine cuts.

Pseudo Guillotine Separability



A group of compartments is said to be pseudo guillotine separable if all the compartments can be extracted using pseudo guillotine cuts and guillotine cuts.

Pseudo Guillotine Separability



A group of compartments is said to be pseudo guillotine separable if all the compartments can be extracted using pseudo guillotine cuts and guillotine cuts.

Formula for Success

Pseudo guillotine separable compartments

$+$

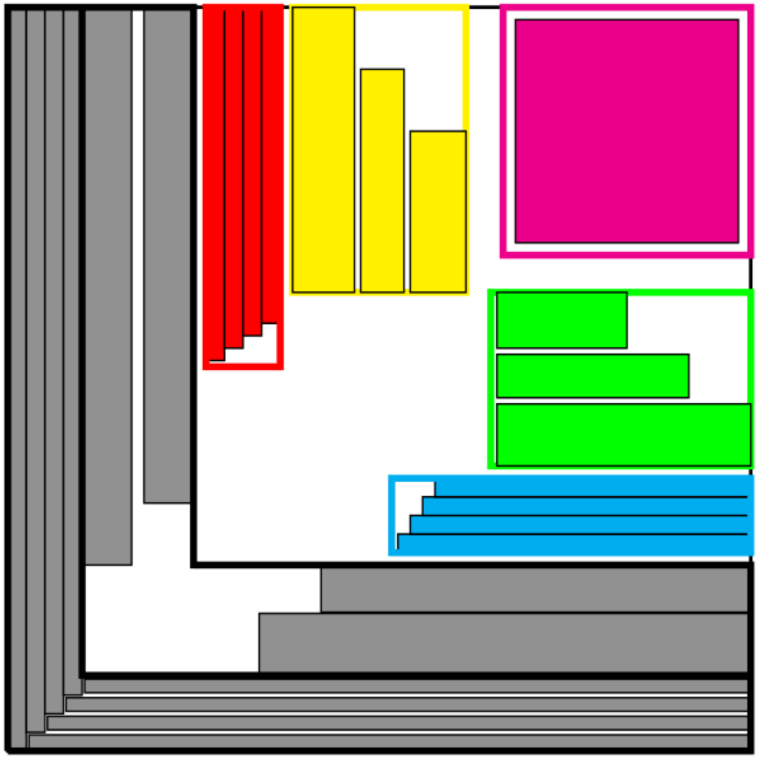
Nicely packed compartments

$=$

Guillotine Separable packing

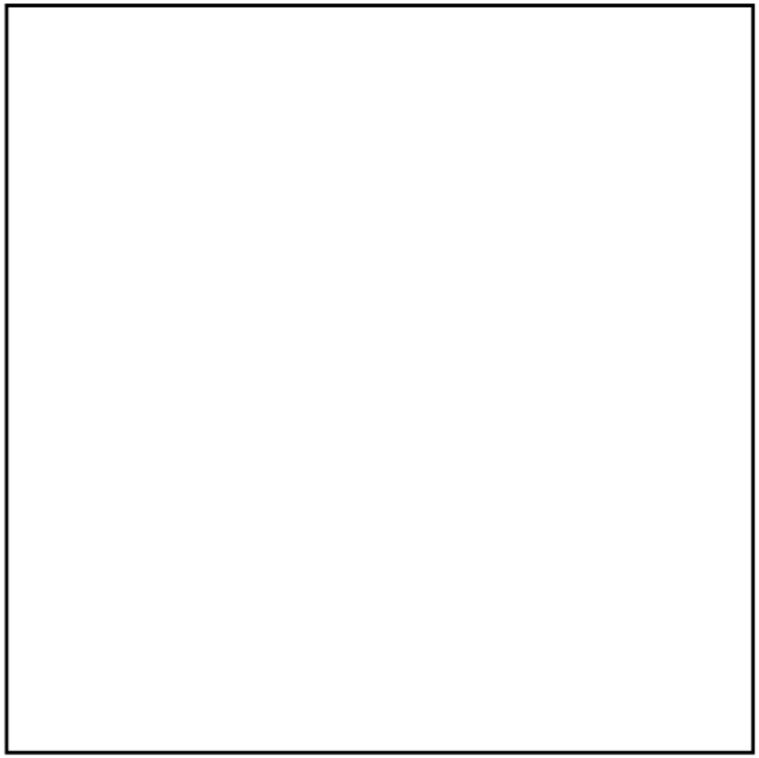
Nice near-optimal structural packing

If we can show the existence of a near optimal nice packing of horizontal, vertical and large rectangles into $O_{\varepsilon}(1)$ pseudo guillotine separable compartments, we can find a near optimal packing in pseudo polynomial time

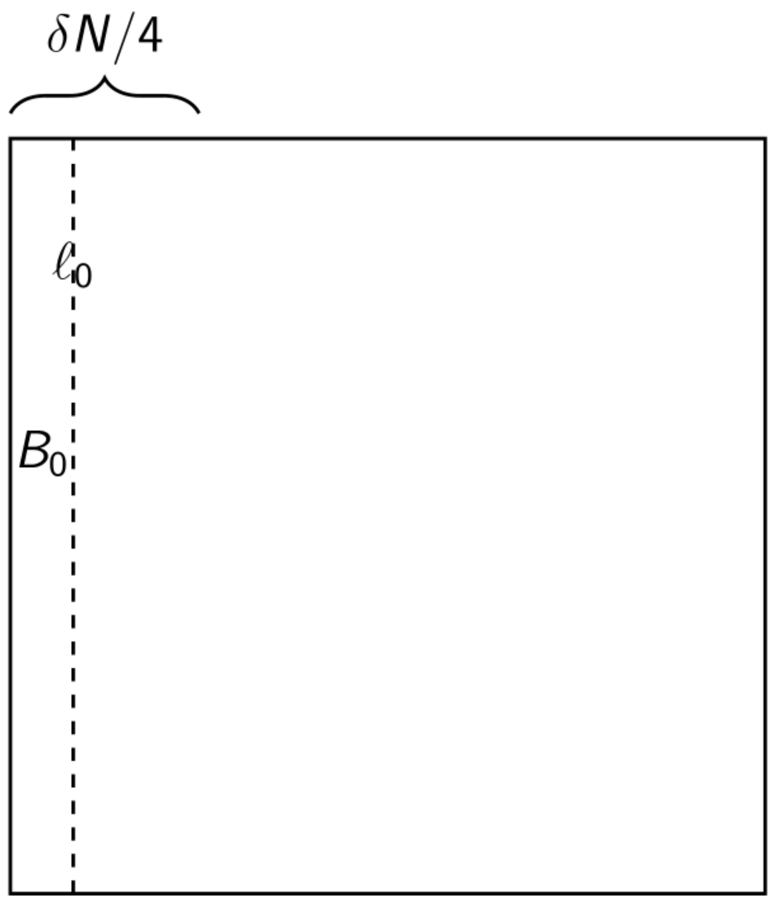


Obtaining a near-optimal structural packing

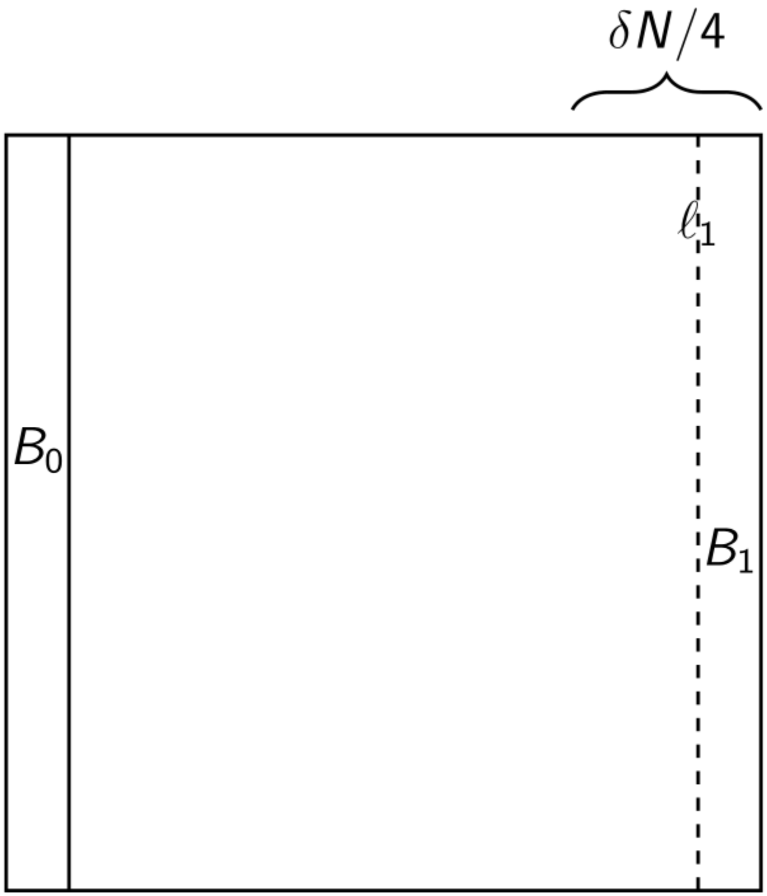
Consider the knapsack with optimal packing. We use the guillotine cuts in this knapsack to obtain a near-optimal structural packing.



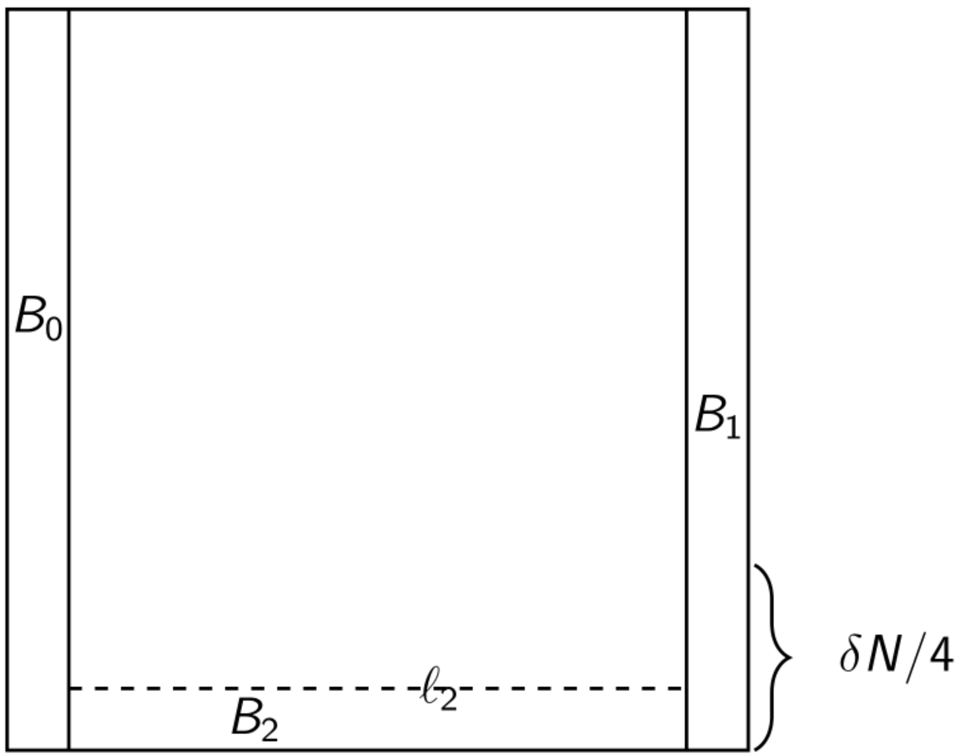
Obtaining a near-optimal structural packing



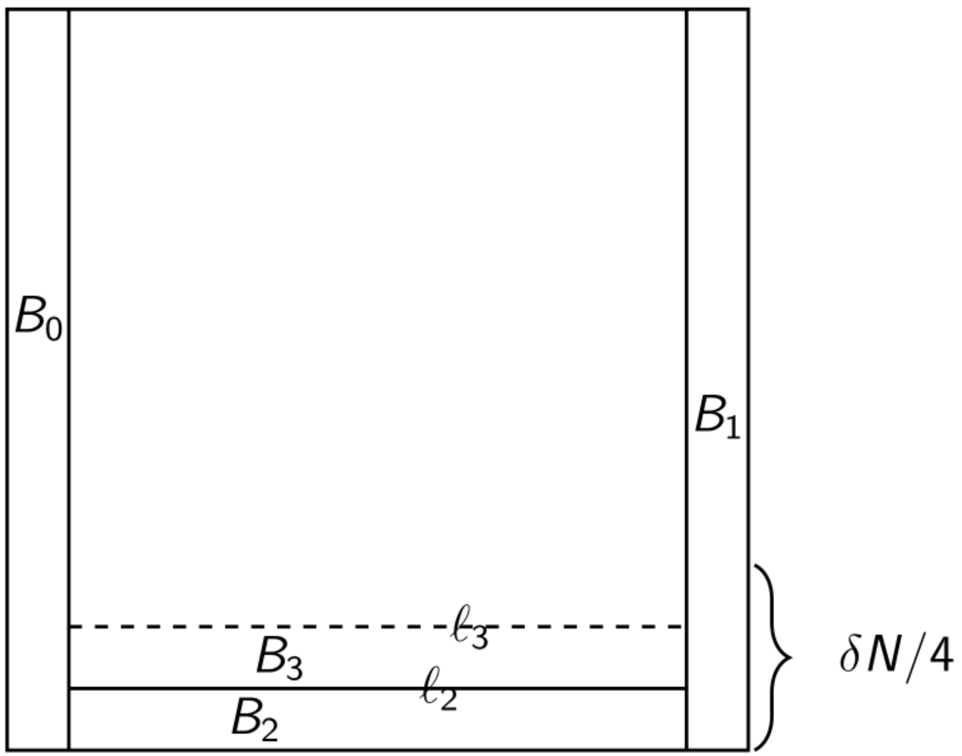
Obtaining a near-optimal structural packing



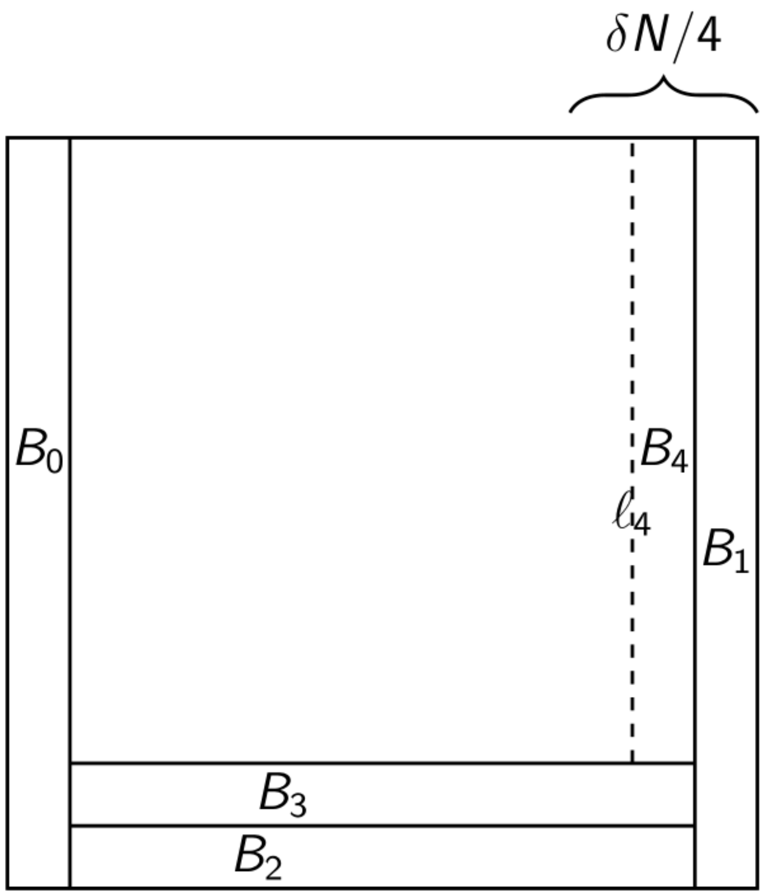
Obtaining a near-optimal structural packing



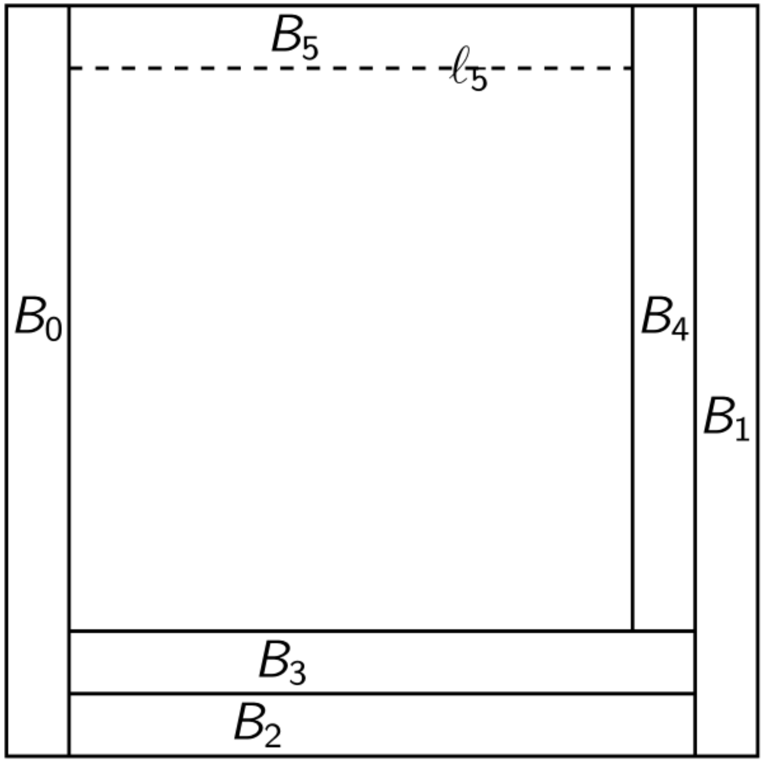
Obtaining a near-optimal structural packing



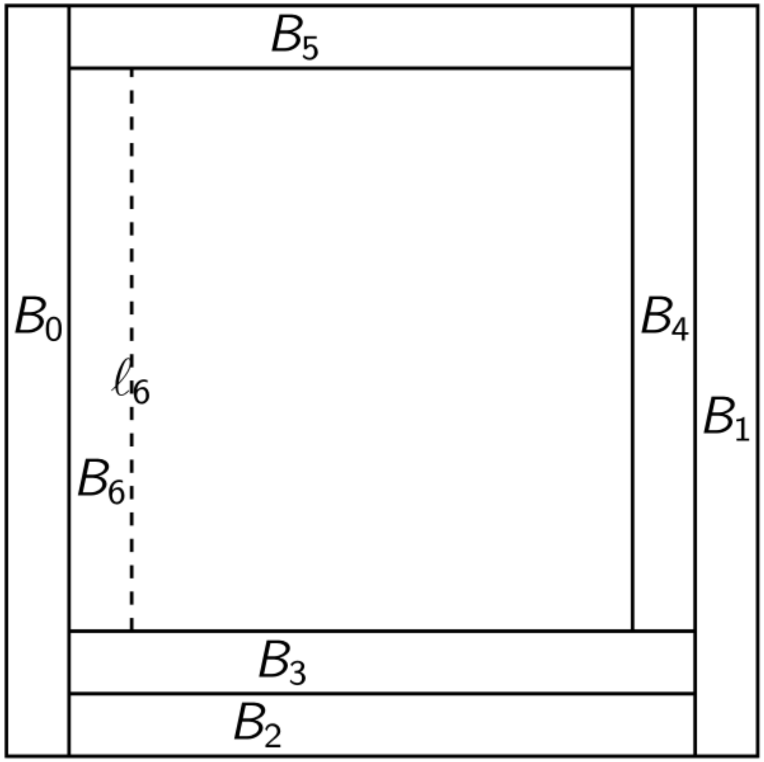
Obtaining a near-optimal structural packing



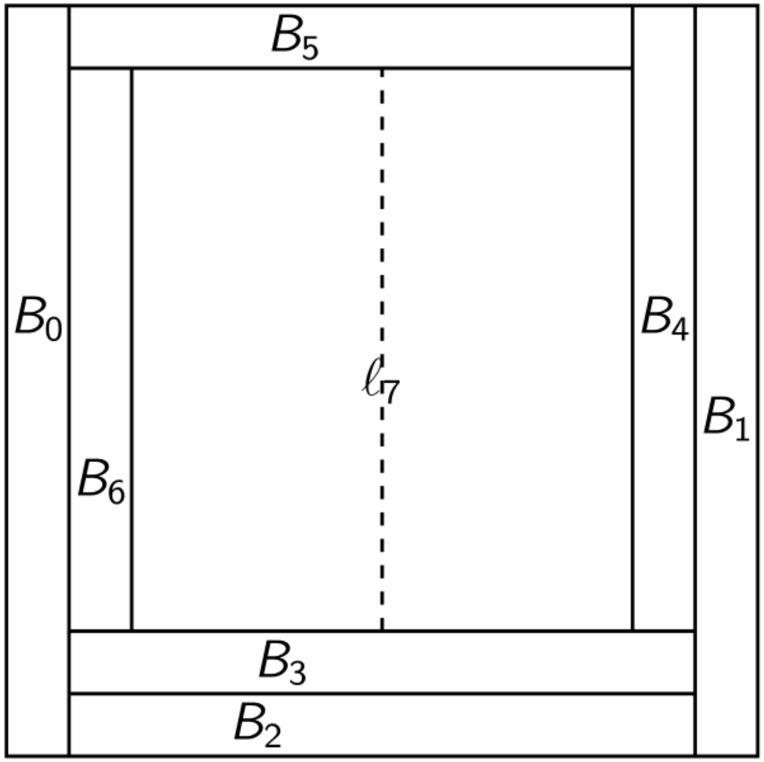
Obtaining a near-optimal structural packing



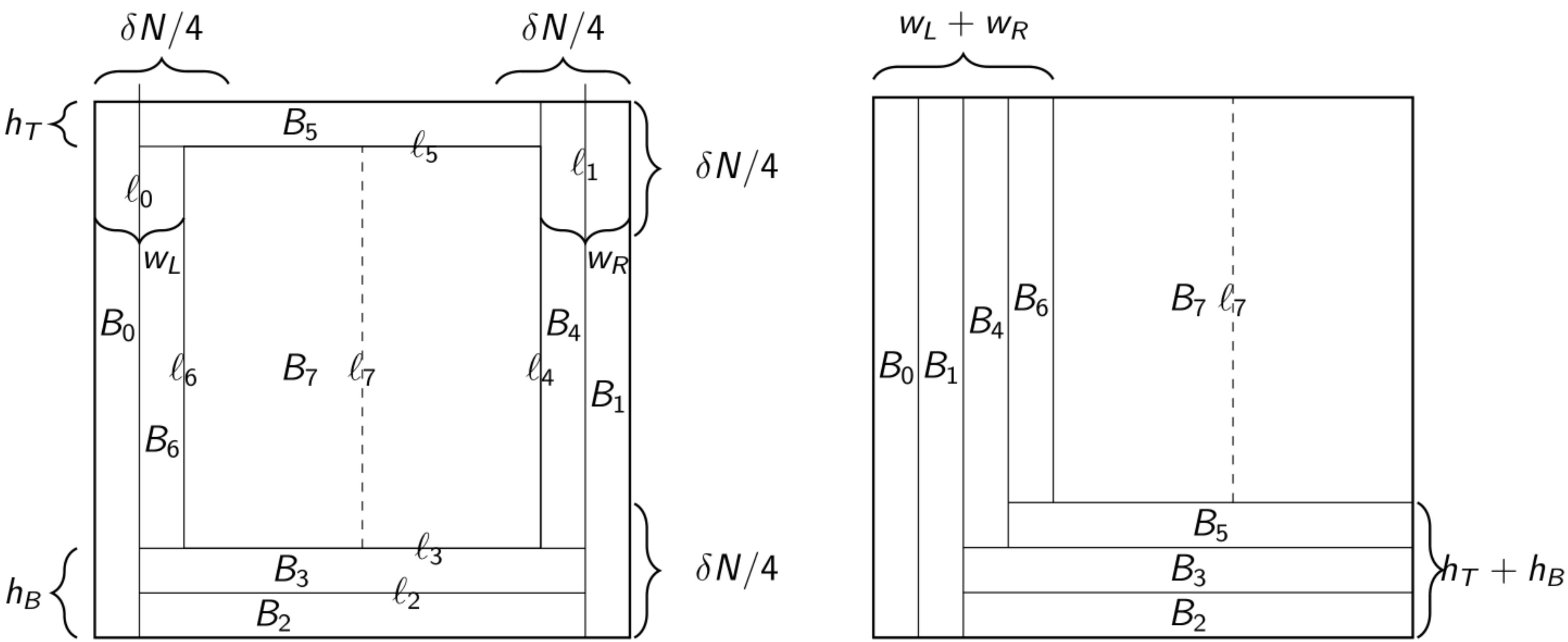
Obtaining a near-optimal structural packing



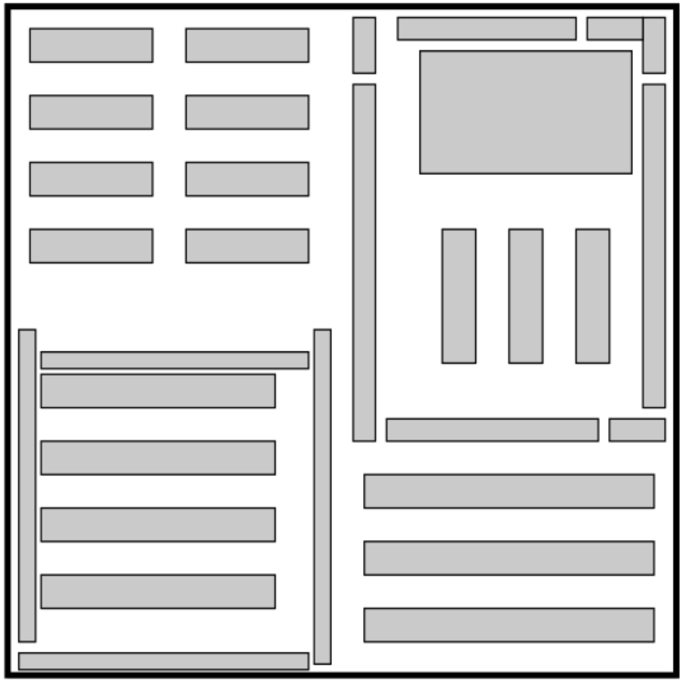
Obtaining a near-optimal structural packing



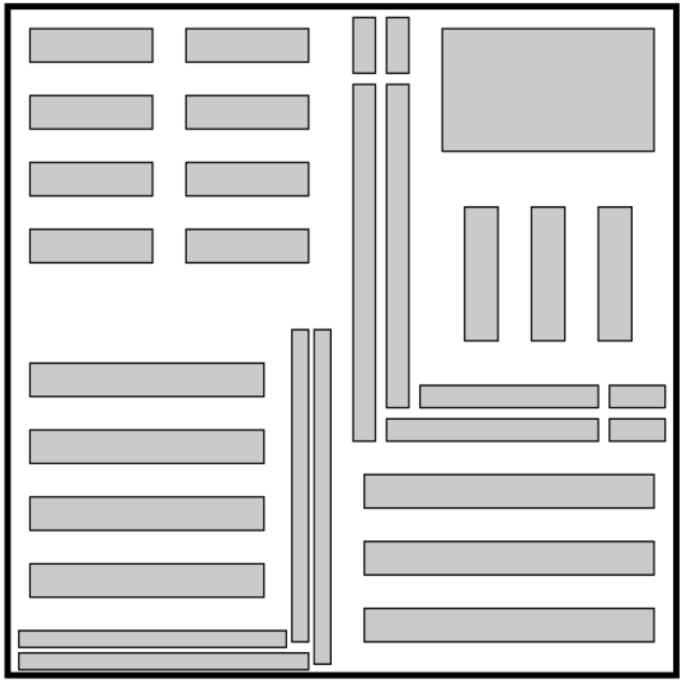
Obtaining a near-optimal structural packing



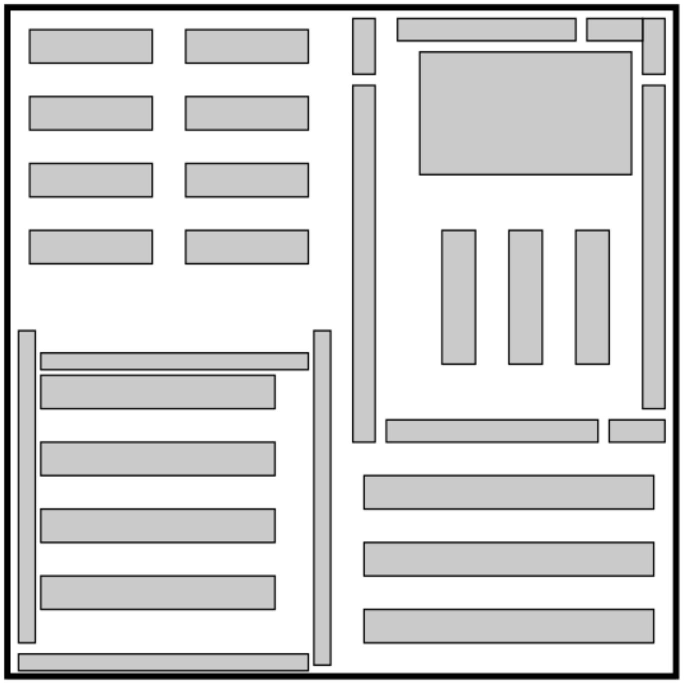
Obtaining a near-optimal structural packing



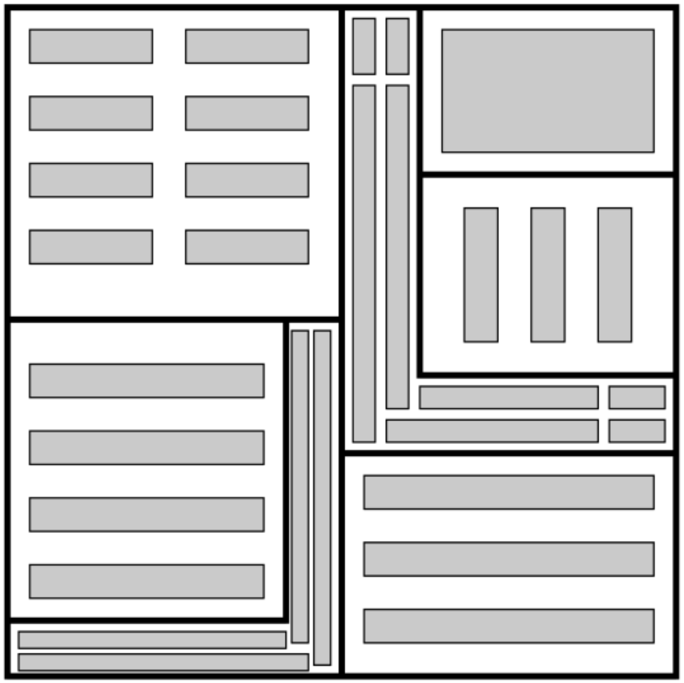
Obtaining a near-optimal structural packing



Obtaining a near-optimal structural packing



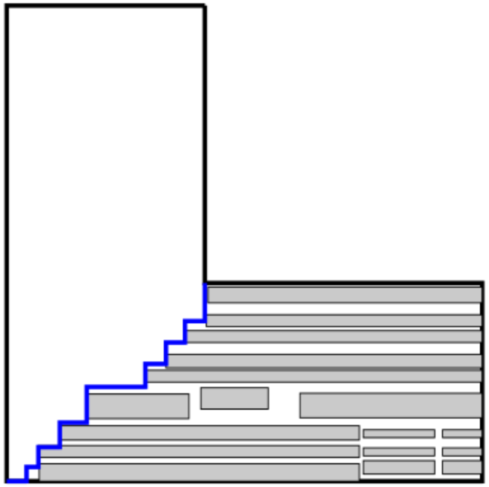
Optimal Packing



Near-optimal Structured Packing

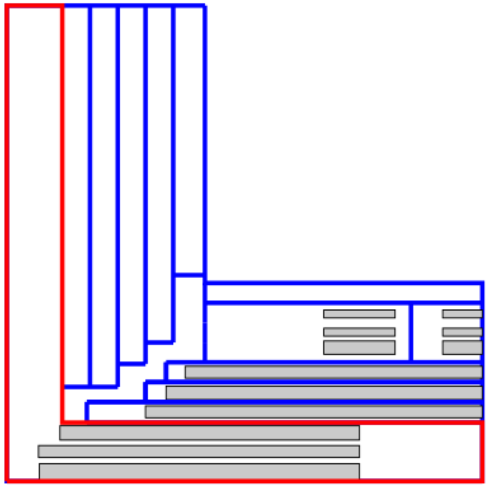
Obtaining Nice packing in Compartments

- Items obtained in our previous structural packing need not be nicely packed
- By removing items of negligible profits and using sophisticated techniques like Shifting Argumentation, Resource Augmentation and Steinberg Packing we can obtain a nice packing of items



Obtaining Nice packing in Compartments

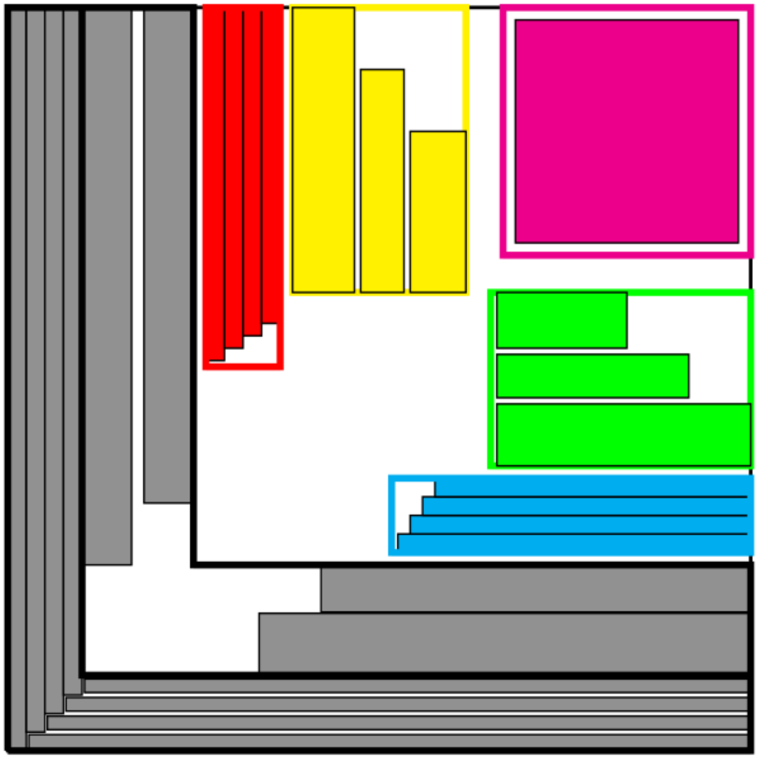
- Items obtained in our previous structural packing need not be nicely packed
- By removing items of negligible profits and using sophisticated techniques like Shifting Argumentation, Resource Augmentation and Steinberg Packing we can obtain a nice packing of items



Nice near-optimal structural packing

We showed the existence of packing s.t :

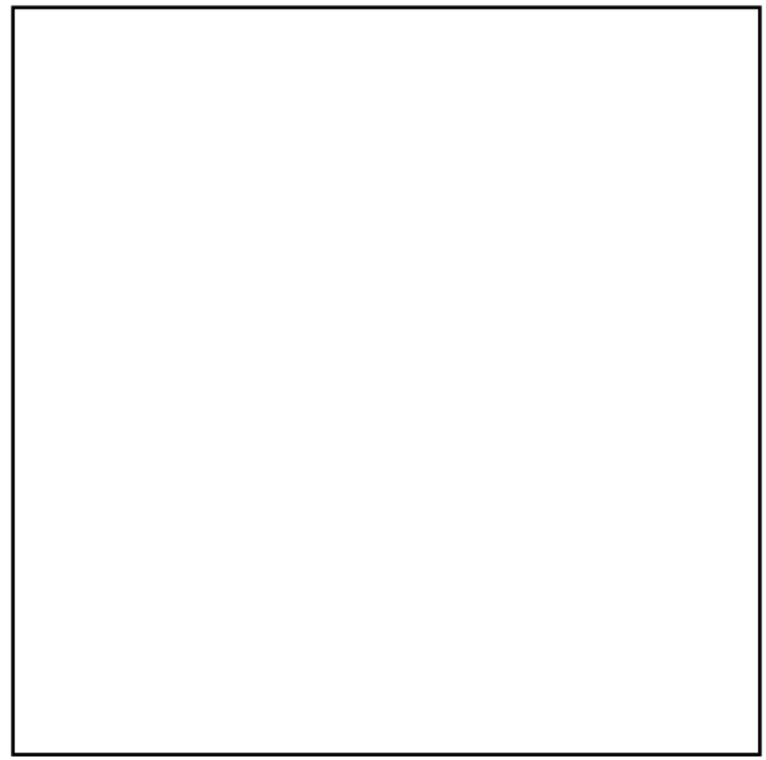
- Packing is Near Optimal
- Items are nicely packed inside the box and **L**-compartments
- The compartments are pseudo-guillotine separable
- The number of compartments is $O_\varepsilon(1)$.



Algorithm Part-1: Guessing the compartments

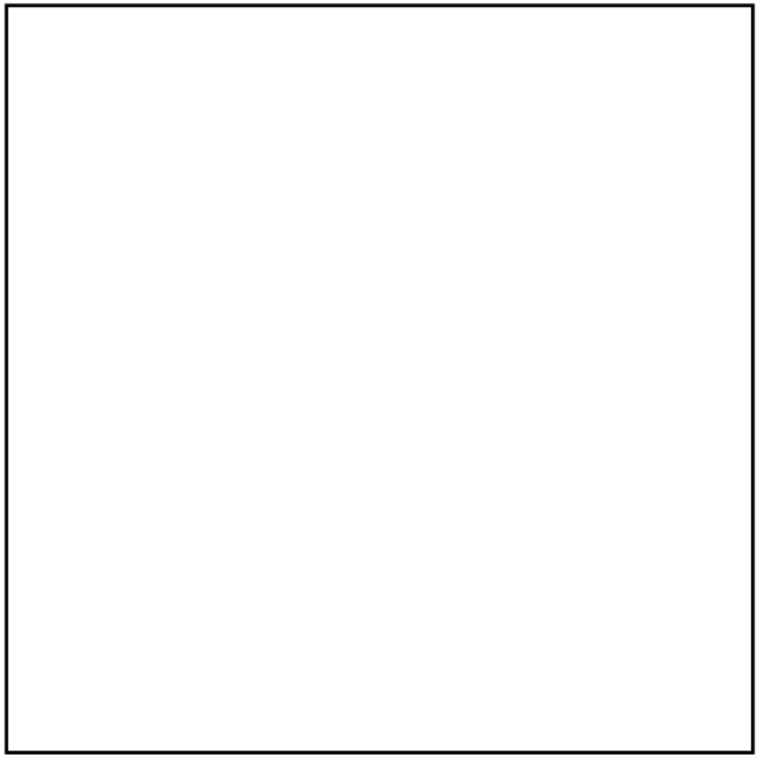
Guessing the compartments in Near-Optimal Nice Packing

- Consider the $N \times N$ knapsack.



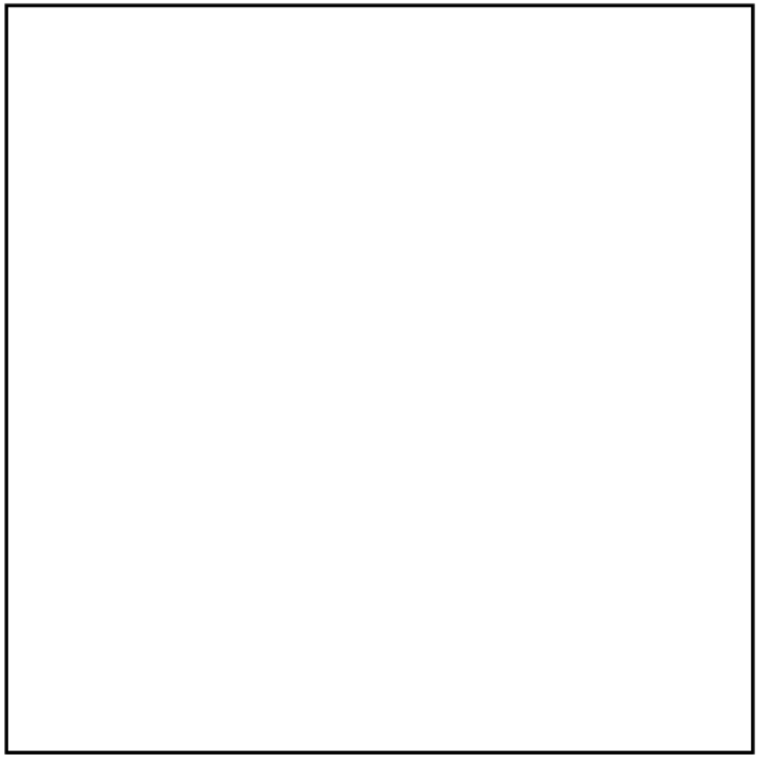
Guessing the compartments in Near-Optimal Nice Packing

- Consider the $N \times N$ knapsack.
- Near-Optimal nice packing has $O_\varepsilon(1)$ compartments.



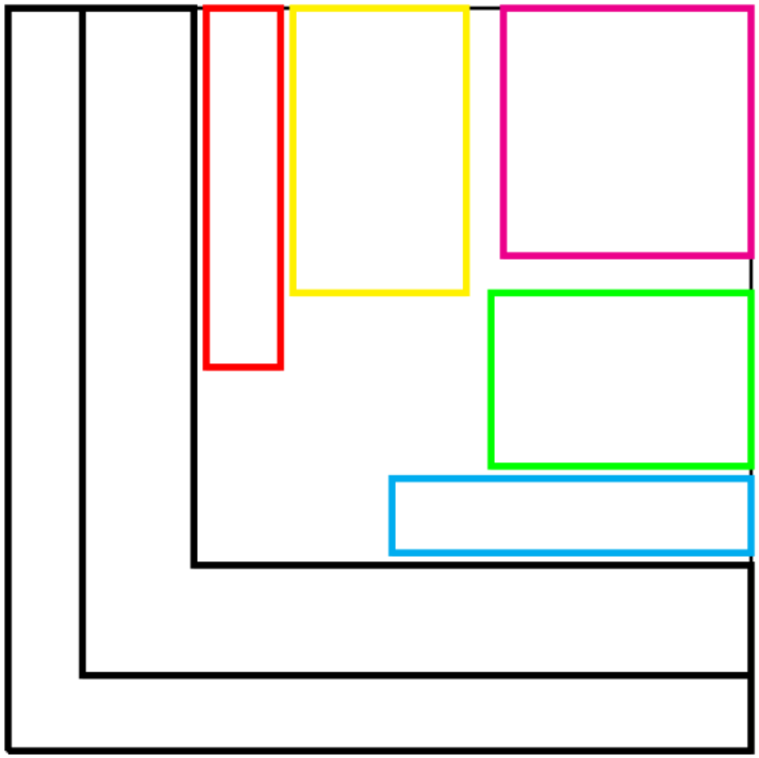
Guessing the compartments in Near-Optimal Nice Packing

- Consider the $N \times N$ knapsack.
- Near-Optimal nice packing has $O_\varepsilon(1)$ compartments.
- There are $N^{O(1)}$ different possible compartments in a knapsack.

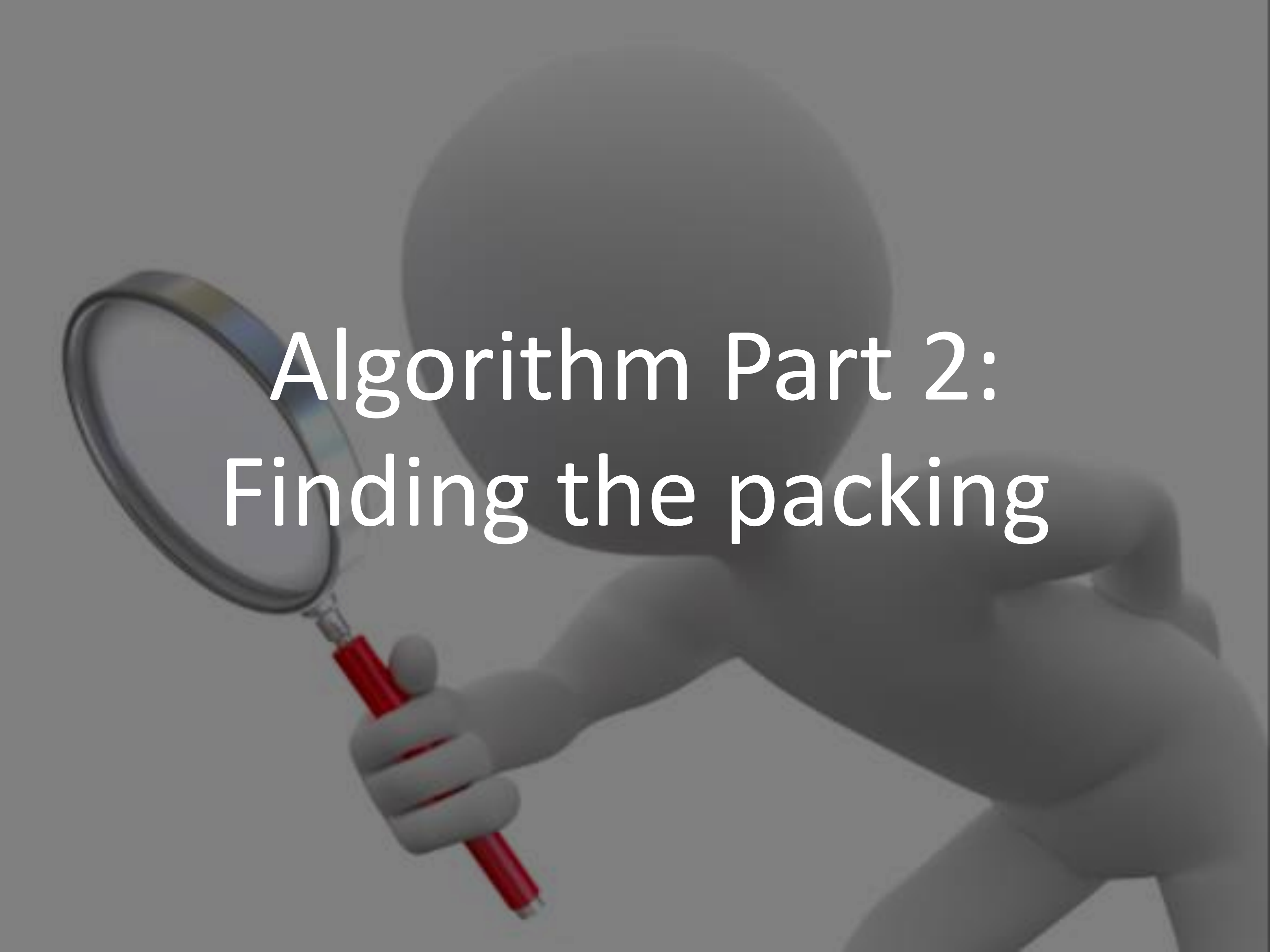


Guessing the compartments in Near-Optimal Nice Packing

- Consider the $N \times N$ knapsack.
- Near-Optimal nice packing has $O_\varepsilon(1)$ compartments.
- There are $N^{O(1)}$ different possible compartments in a knapsack.
- So we can guess the compartments in Near-Optimal Nice Packing in time $N^{O_\varepsilon(1)}$.

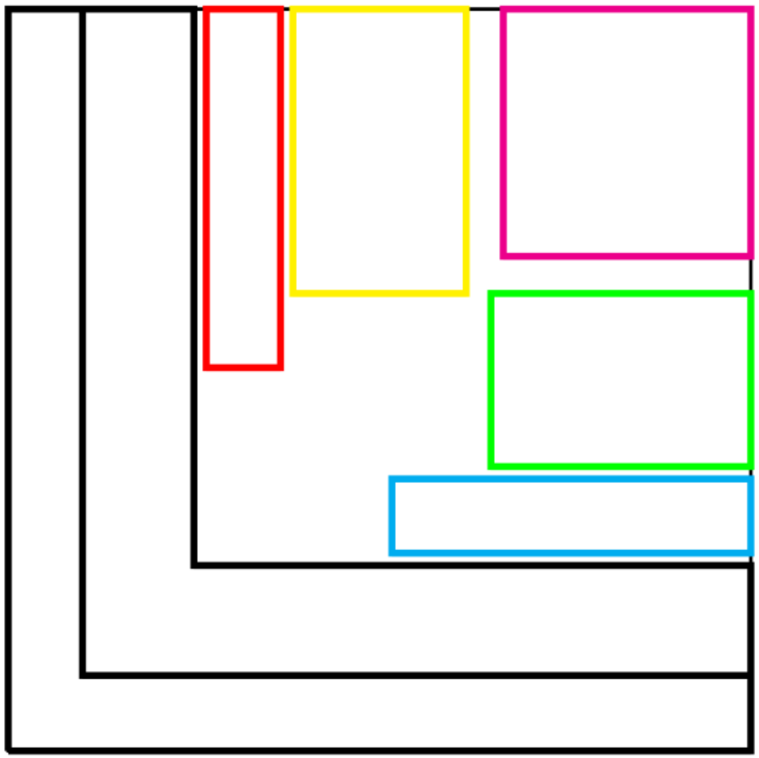


Algorithm Part 2: Finding the packing



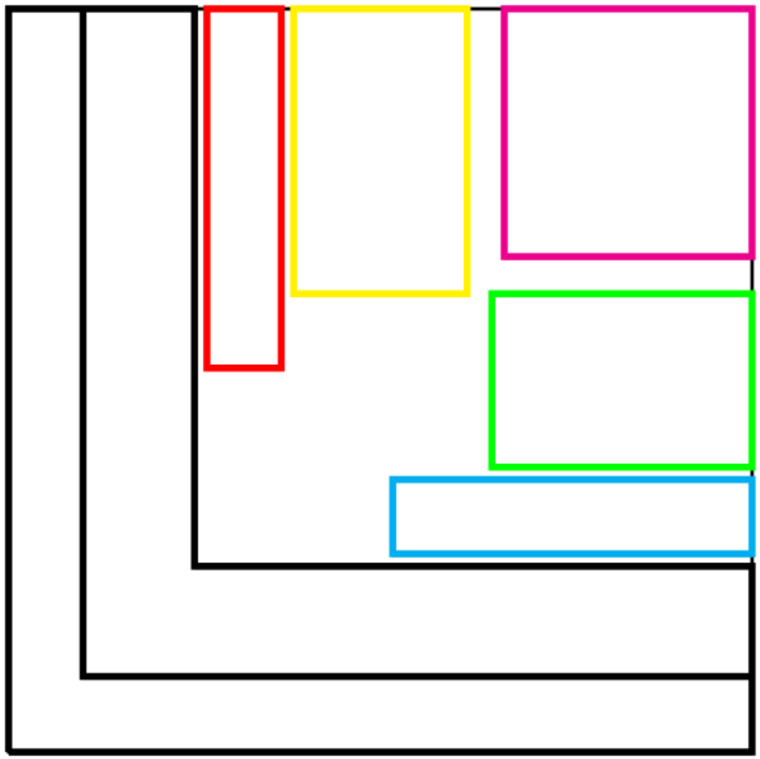
Finding the Near-Optimal Packing in the Guessed compartments

In time $(nN)^{O_\varepsilon(1)}$, we assign a near-optimal subset of items to the guessed compartments by using an algorithm adapted from recent work by Galvez et al. (SoCG'21).



Finding the Near-Optimal Packing in the Guessed compartments

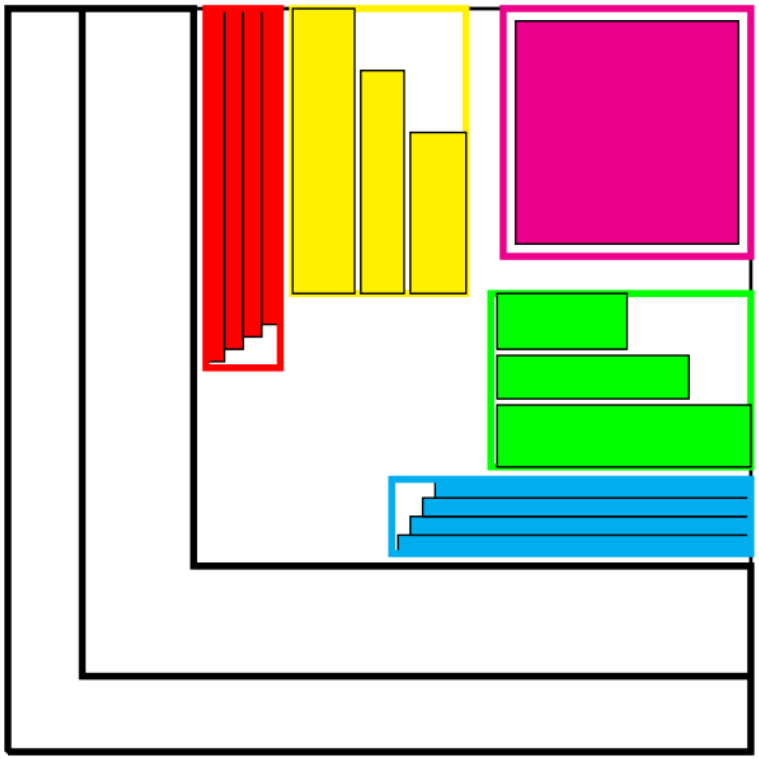
We now pack the items as follows:



Finding the Near-Optimal Packing in the Guessed compartments

We now pack the items as follows:

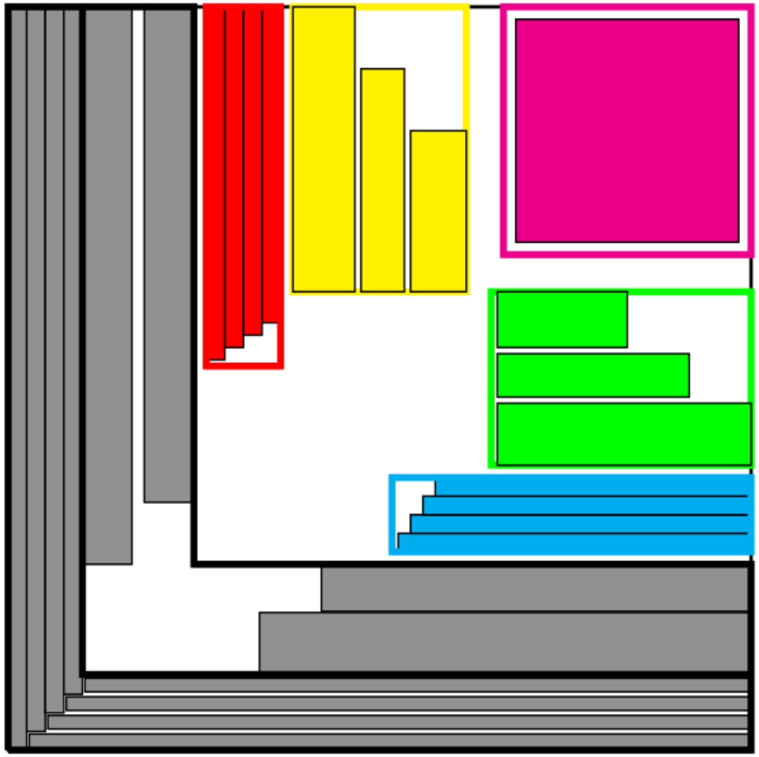
- Box-compartment: Use Pseudo-polynomial time algorithm for 1D Knapsack



Finding the Near-Optimal Packing in the Guessed compartments

We now pack the items as follows:

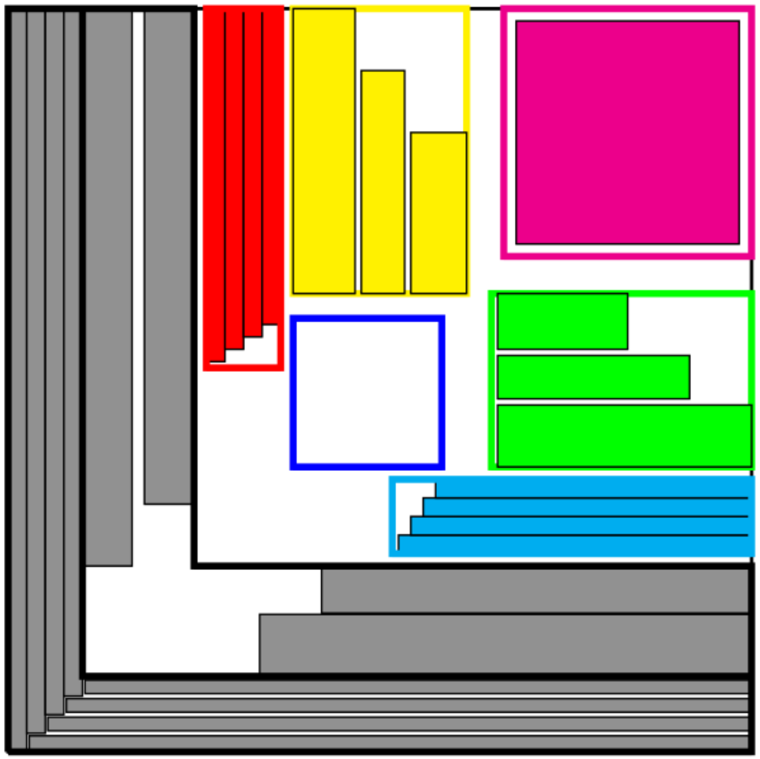
- Box-compartment: Use Pseudo-polynomial time algorithm for 1D Knapsack
- **L**-Compartment: Use Pseudo-polynomial time algorithm for L-packing by Galvez et al. (FOCS'17)



Finding the Near-Optimal Packing in the Guessed compartments

We now pack the items as follows:

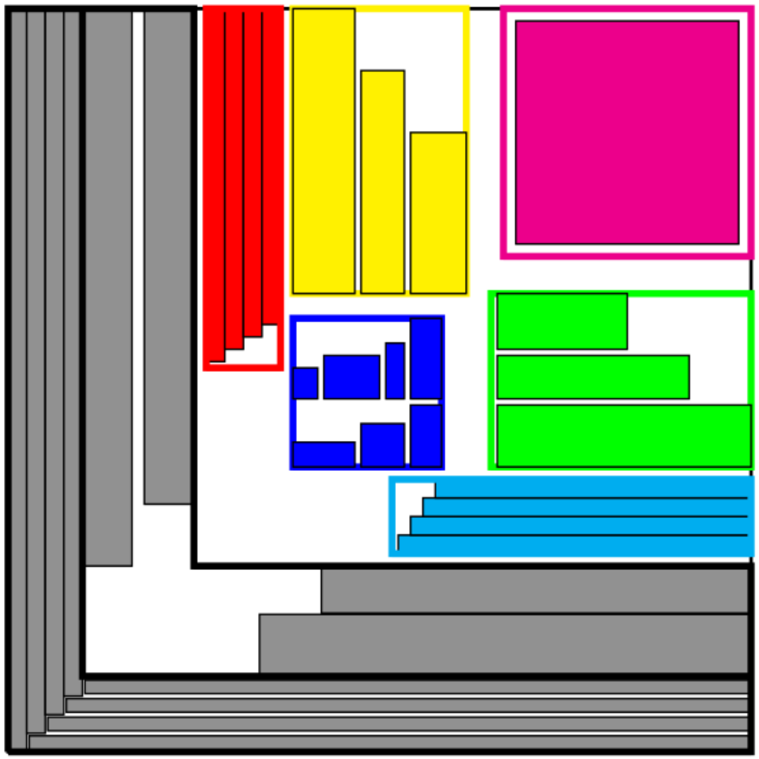
- Box-compartment: Use Pseudo-polynomial time algorithm for 1D Knapsack
- **L**-Compartment: Use Pseudo-polynomial time algorithm for L-packing by Galvez et al. (FOCS'17)
- Small Items: Find Guillotine Separable empty regions



Finding the Near-Optimal Packing in the Guessed compartments

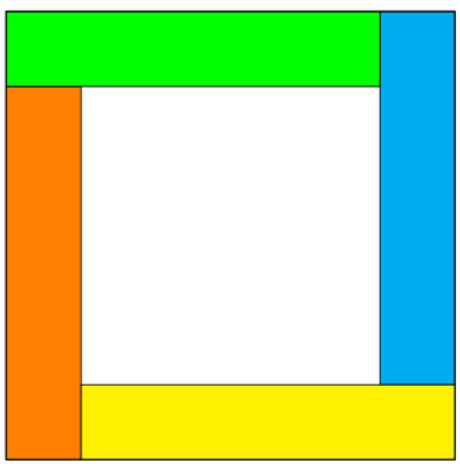
We now pack the items as follows:

- Box-compartment: Use Pseudo-polynomial time algorithm for 1D Knapsack
- **L**-Compartment: Use Pseudo-polynomial time algorithm for L-packing by Galvez et al. (FOCS'17)
- Small Items: Find Guillotine Separable empty regions
- Small Items: Pack using Next Fit Decreasing Height

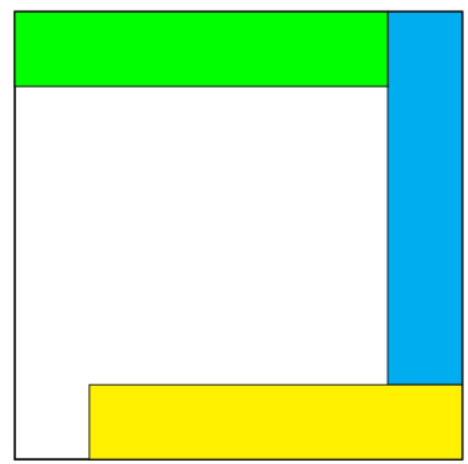


Open Problems

- Is there a PTAS or QPTAS for 2D Guillotine Knapsack ?
- Is there a PTAS or PPTAS or QPTAS for 2D Geometric Knapsack ?
- Prove the following conjecture: the worst-case ratio between the optimal 2D geometric packing and optimal 2D guillotine separable packing is $4/3$



Optimal 2D geometric Packing



Optimal 2D guillotine separable Packing


THANK YOU