On Guillotine Separable Packings for the Two-dimensional Geometric Knapsack Problem

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1. Introduction

Guillotine Cut: An end to end axis parallel cut dividing a plane.
Guillotine Separable Packing: Each item can be cut out by a sequence of guillotine cuts.

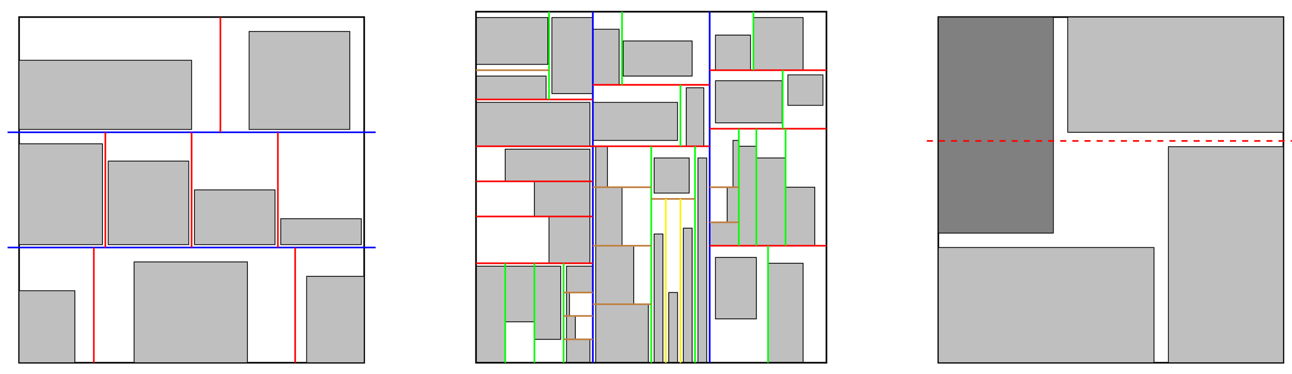


Figure 1: All except rightmost figure are guillotine separable packings.

2-D Knapsack (2GK):
Given:
- n-rectangular axis-parallel items; each item (say i) with an integral height ($h(i)$), width ($w(i)$) and profit ($p(i)$),
- $N \times N$ square knapsack where $N \in \mathbb{N}$

Goal: Find a maximum profit non-overlapping packing of subset of items.
Variants:
- With (2GK(R)) or without rotation (2GK)
- Cardinality case (equal profits) (2GK-C),
- Weighted Case (2GK).

2-D Guillotine Knapsack (2GGK). Goal: Find maximum profit non-overlapping, *guillotine separable* packing of a subset of items.

2. Prior Works

Previous known approximation of 2GGK:
- $3+\epsilon$ [Jansen-Zhang,SODA'04](for all cases),
- QPTAS [Abed et al, APPROX'15]

Previous known approximation of 2GK [Galvez et al. FOCS'17]:
- Cardinality Case:
  - $1.89+\epsilon$, $1.5+\epsilon$ (w and w/o rotations)
- Weighted Case:
  - $1.72+\epsilon$, $1.33+\epsilon$ (w and w/o rotations)

3. Previous Techniques

Container Packing: *Container* is an axis-aligned rectangular region such that either it contains (1) One large item, (2) Items packed as horizontal stack or vertical stack or, (3) Items that are very small in both dimensions.

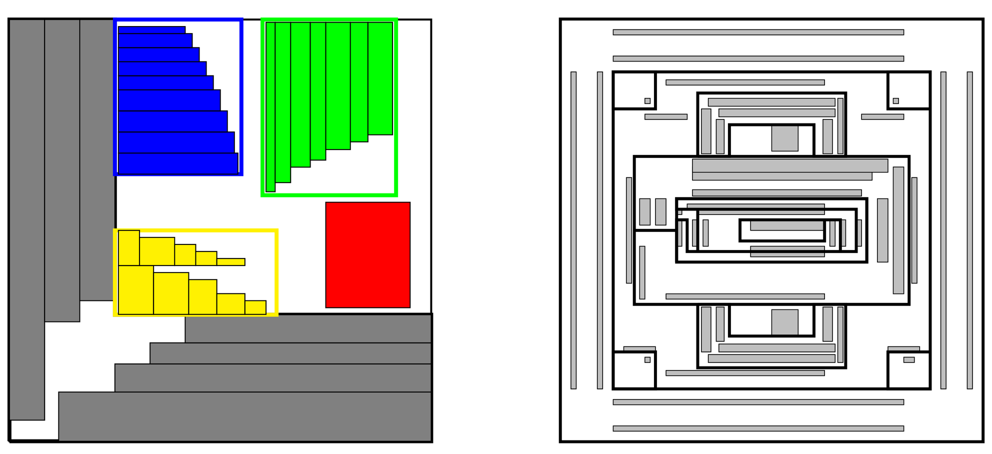


Figure 2: [Left] Shows container packing. [Right] Corridor Decomposition

Corridor Decomposition: Partition the knapsack using a constant number of lines with constant (in fact at most 2) bends, such that the intersected rectangles have small weight.

4. Our Techniques

We obtain $(1+\epsilon)$-pseudo-polynomial algorithm (PPTAS) for both 2GGK and 2GGK(R), where input numbers are bounded by $poly(n)$, using:
- **Structural Lemma:** Existence of near-optimal "nicely" structured solutions. Define two type of compartments; **L** and **box**. Packing is *nice* i.e. inside each compartment, the items are placed such that all horizontal items are simply stacked on top of each other, all vertical items are placed side by side, and all small items are packed greedily with the NFDH algorithm. The packing is *pseudo-guillotine separable* i.e. each compartment (pseudo-item) is guillotine separable from the others.
- **Assigning Items:** Next, guess the nice packing in $(nN)^{O_\epsilon(1)}$ time. Handle packing of small, large and skew items using different techniques.

5. Structural Lemma

Recursively consider the guillotine cutting sequence in OPT as long as one of the subplanes is small enough. Restructure the subplanes using flipping and mirroring techniques; preserving guillotine property; to obtain a **L** and a **box** compartment as shown in Figure 3. Compartments can be separated from each other using a sequence of guillotine cuts.

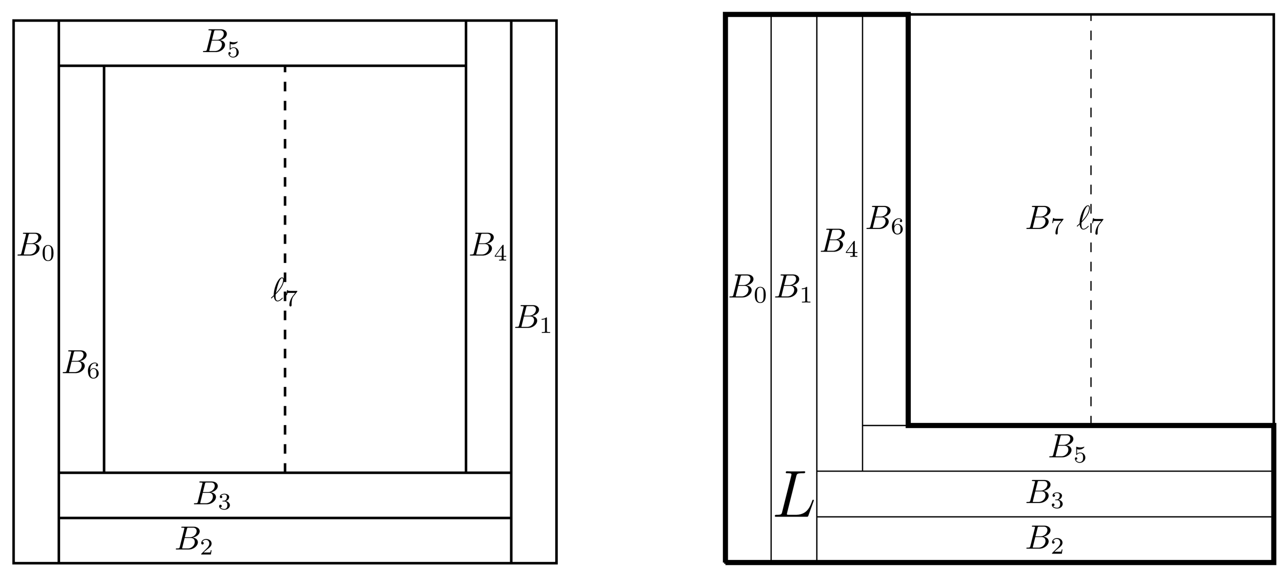


Figure 3: Step1: Consider Guillotine Cuts recursively. Step2: Rearrange to obtain a L

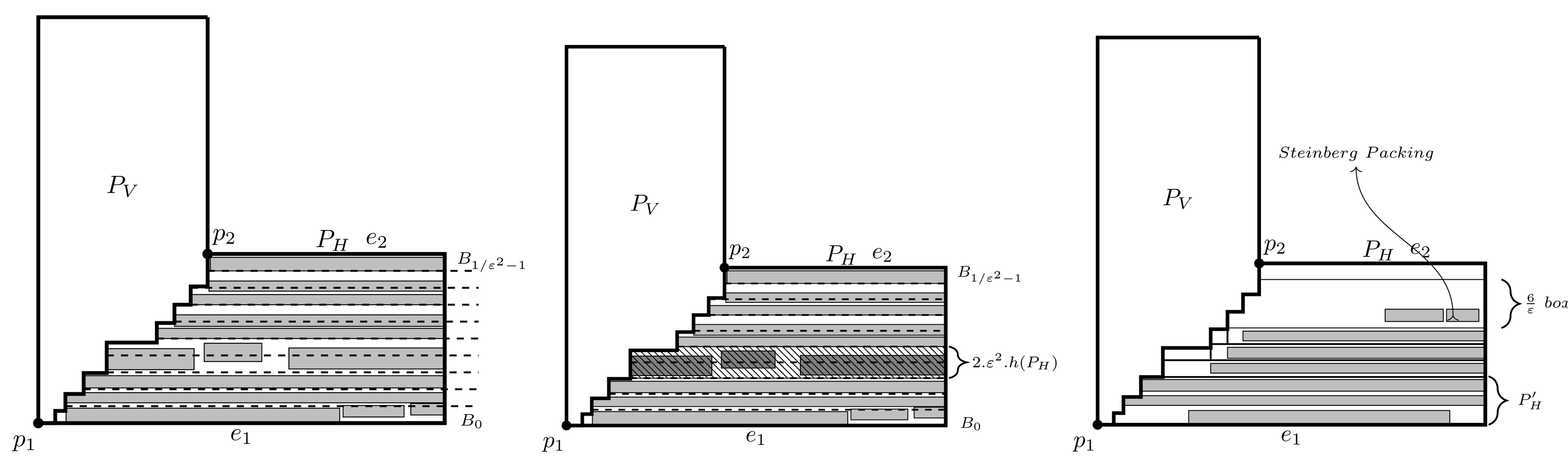
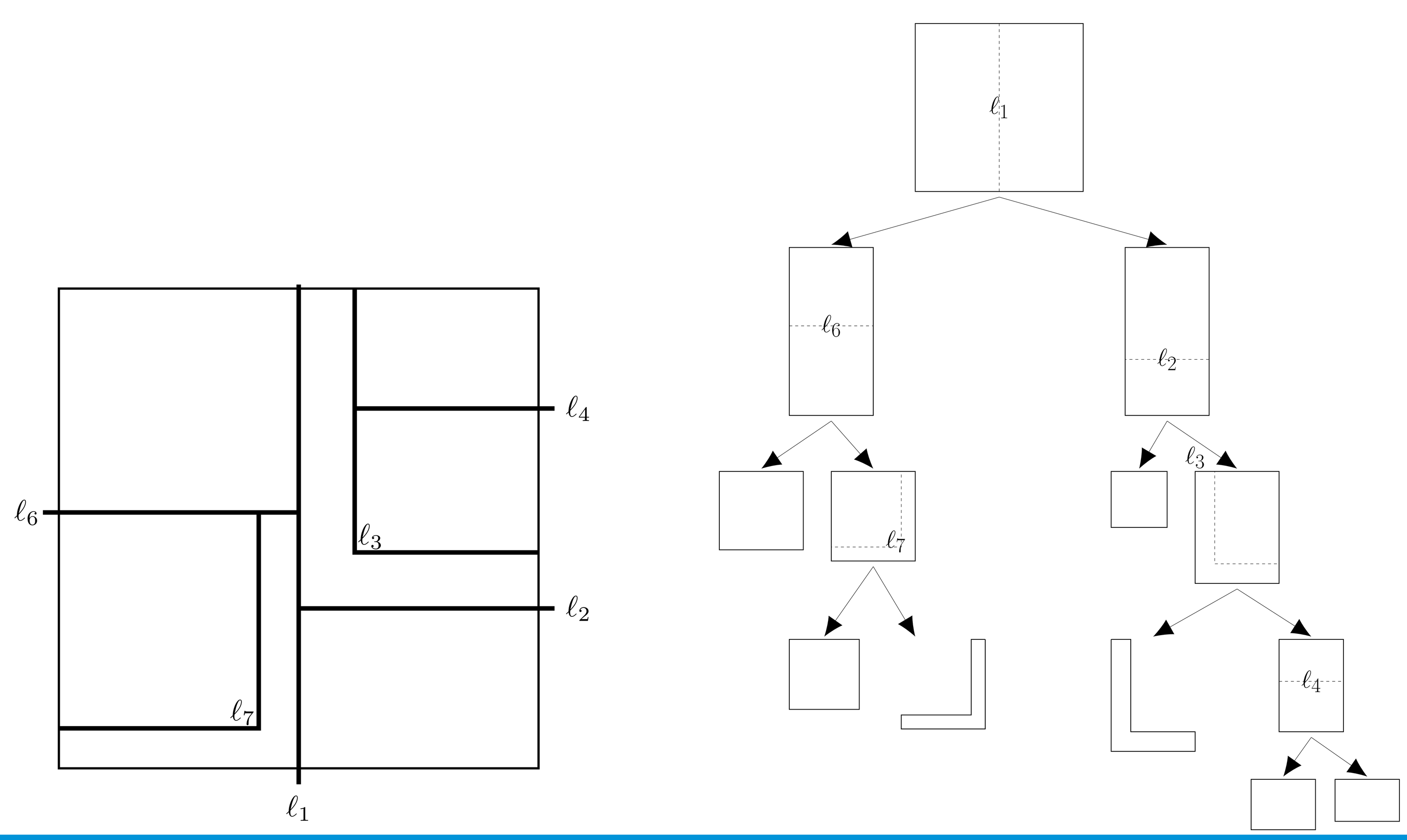


Figure 4: Processing done to obtain nice packing of L-compartment.

Process each **L** and **box** compartment to obtain a nice packing of items. Divide the legs of L into strips and remove the stripes to shift and accommodate problematic items inside the **L** (shown in Figure 4).

Packing obtained is *pseudo-guillotine separable* i.e. the compartments (when considered as pseudo-items) are guillotine separable using a sequence of guillotine cuts as shown in figure on the right.



6. Final Results

Main algorithm works as follows:

- First, guess the $O_\epsilon(1)$ compartments in time $(nN)^{O_\epsilon(1)}$.
- Then place items nicely inside box compartments using NFDH (Next-Fit-Decreasing-Height) algorithm. For packing items in L-compartments we use a $(1+\epsilon)$-approximation algorithm which is a slight adaptation of a recent pseudo-polynomial time algorithm on 2GK by Galvez et al. [SoCG'21]
- The resulting solutions use up to $\Theta(\log(nN))$ stages of guillotine cuts. We also give the lower bound $\Theta(\log(nN))$ stages for any $(2-\epsilon)$ approximation (as shown in figure 5).

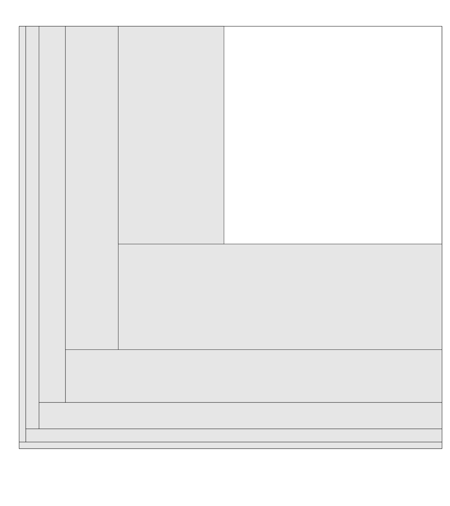


Figure 5: Hard Example with $\Theta(log(nN))$ stages.