The Art Gallery Problem: Status and Perspectives

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Surveillance in art galleries is one of the celebrated problems in graph theory and computational geometry that deals with visibility issues. In this survey paper, we study the art gallery problem both from the perspectives of historical results and recent advances. We highlight major bounds on a number of complexity and existential questions that have been currently answered for the classical variants as well as for the orthogonal variants of the problem. Following the fact that the problem is NP-Hard (recently it has proven to be $\exists R - Complete$ by Anna Adamaszek et al. [AAM18]), we present developments in the problem when seen through lenses of Parameterized and Approximation Algorithms.

CCS Concepts: • **Theory of computation** → *Design and analysis of algorithms*; **Approximation algorithms analysis**; **Parameterized complexity and exact algorithms**.

Additional Key Words and Phrases: visiblility, simple polygons, approximation algorithms, parameterized algorithms, triangulation

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1 INTRODUCTION

The twenty-first century has witnessed many revolutionary technologies such as GPS navigation, area surveillance, automated vehicles, robotic mobility, assisted living, and smart-home devices to name a few. The common question that comes up while designing the these technologies is "what if in one way computers could learn to localize themselves in a closed environment like a room or a dining table or probably a concealed room in a bank, so as to secure/guard the whole field?". All of this can be summarized by the Art Gallery Problem. The problem dates long back to 1973, when Victor Klee posted the problem statement titled "The Art Gallery Problem" to Vaclav Chvàtal. The question Klee asked was "Consider an art gallery, what is the minimum number of stationary guards needed to protect the room?" In geometric terms, the problem was stated as: "Given an n-vertex simple polygon, what is the minimum number of guards to see every point of the interior of the polygon?" Chvàtal, who received the problem, was able to prove a bound of $\lfloor \frac{n}{3} \rfloor$ for the sufficient (as well as necessary) number of guards to guard the gallery. Consequently the Art Gallery Problem has become arguably one of the most well-known problems in discrete and computational geometry. Numerous research works have been published on the subject. O'Rourke's early book from 1987 [O'r87] has more than 2,000 citations, and each year, top conferences publish new results on the topic. The problem was proven NP-Hard in 1994-95 but couldn't be proven NP-complete. Until very recently in 2017, Abrahamsen et al. proved the problem to be $\exists R$-complete. These results were preceded by Abrahamsen's proof which established that sometimes irrational guards are necessary for guarding the art gallery. This result posed the potential hardness of the problem.

Years of work on the problem and the requirement of specific constraints in robotics and other practical applications, the problem developed multitude of variants based on the type of guards allowed, region to be guarded or the type of polygon. The problem has also been studied in depth using numerous tools of theoretical computer science and computational geometry ranging from exploring bounds on the sufficient number of guards to approximation and parameterized algorithms. This paper is an attempt to survey and accumulate all major results developed into these particular variants of Art Gallery Problem.

1.1 Paper Structure

The contributions of paper are essentially a highlight of the current results on the problem as a literature survey. We initialize with introducing the required terminology in Section 2.1. In Section 2.2, we formally define the

Art Gallery Problem and the variants we will be considering for our literature survey. Results in the paper are sectioned into essentially 4 parts. In Section 3 we establish NP-Hardness of the problem under various constraints. We present that the problem has not yet been proven to be $NP - Complete$, and present the result of $\exists R - Completeness$ of the problem. To emphasise on the potential hardness of tractability of the problem, we present a proof that irrational guards are sometimes necessary in polygons to optimally guard the given region (Section 3.1). In Section 4 we exploit bounds on numerous variants of the problem including Classical Art Gallery Problem, Orthogonal Art Gallery Problem (resp. Section 4.1 and Section 4.2). The section shows results on variants including edge guard, vertex guard, holes in polygons.

In Section 5, we exploit the NP-Hardness of the problem and highlight 2 major approximation results with $O(logn)$, $O(logOPT)$-approximation factor for both vertex guard and edge guard problems (resp. Section 5.2, Section 5.3). Then we deviate from the classical problem and consider a variant of the problem by relaxing the visibility criteria and present a 6-approximation result by Ghosh et al. on visibility polygons (Section 5.4). Next we extend our approach to parameterized algorithms in Section 6 showing hardness results w.r.t the natural parameter for the variants edge and vertex guard using the ETH-Hypothesis. We then present an FPT by Lokshtanov et al. w.r.t reflex vertices (Section 6.2). We conclude the survey with a set of open problems and potential future directions in AGP in Section 8.

List of Figures

2 PRELIMINARIES

In this section, we will define the terms appearing frequently in the paper.

2.1 General Definitions

Here we define the terms which will be useful in the later section -

Definition 1 (Simple Polygon). *A polygon that does not intersect itself and has no holes.*

Definition 2 (Visibility). *Two points u and v in a polygon P are said to be visible if the line segment joining u and v lies entirely inside P.*

Definition 3 (Fan polygon). *A simple polygon is called a fan if there exists a vertex that is visible from all points in the interior of the polygon.*

Definition 4 (Convex Polygon). *A convex polygon P is a simple polygon such that for every two points p and q on the boundary (or interior) of P, no point of the line segment pq is strictly outside P.*

Definition 5 (Monotone Polygon). *A polygon P in the plane is called monotone with respect to a straight line L, if every line orthogonal to L intersects P at most twice.*

Definition 6 (Reflex Vertices). *A vertex V of a polygon is a reflex vertex if its internal angle is strictly greater than* π.

Definition 7 (Polygon with holes). *If the boundary of P consists of two or more cycles, then P is called a polygon with holes. Otherwise, P is called a simple polygon or a polygon without holes. Alternatively, given a polygon P and a set of m disjoint polygons* $P_1; ...; P_m$ *contained in the interior of P , we call the set* $P - \{P_1 \cup \ldots \cup P_m\}$ *a polygon with holes. In this case, we say that P has m holes.*

Definition 8 (Planar Graph). *A graph which can be drawn on a plane such that no two edges intersect is a planar graph.*

Definition 9 (Triangulated Graph). *A planar graph G is said to be triangulated (also called maximal planar) if the addition of any edge to G results in a non-planar graph.*

Definition 10 (n-triangulation). *A planar graph with n vertices such that one of its faces is bounded by an n-gon and each of the remaining faces is bounded by a triangle.*

Definition 11 (Fan triangulation). *A k-triangulation will be called a fan triangulation if one of its vertices meets all of its k - 3 non-boundary edges.*

2.2 Art Galleries and Variations

Art Gallery Problem has been studied considering a lot of variations. We define the problem in the most general way as follows:

ART GALLERY PROBLEM (AGP)

Input : A simple polygon P, (possibly infinite) sets G and C of points within P
Goal : Find the minimum set $S \subseteq G$ such that every point in C is visible to at least 1 guard in S.

We can similarly define the decision version of the problem, with an additional input integer k. The goal of decision version is to decide whether there exists a set $S \subseteq G$, such that $|S| \leqslant k$ and every point in C is visible to at least one guard in S.

2.2.1 *Variations of guards.*

Points within polygon P (including the boundary) can be broadly classified into Vertex (vertices of P), Point (all points within P) and the Boundary (all points on the boundary of P). Guard set can also be defined as Mobile where they are allowed to move along closed line segments totally contained in a polygon P. We can restrict the mobile guards to move along an edge and obtain another placement for guard set G along Edge. This gives to rise to variations of AGP, which can be defined as XYG (X, Y guard art gallery) where $G = X \subseteq \{Vertex, Point, Boundary, Edge\}$ and $C = Y \subseteq \{Vertex, Point, Boundary\}$. We can simplify the notation where the points to be guarded i.e. C is the set of all Points in P, and denote it simply by XG (X Guard Art Gallery). We elaborate the possible guard placements as follows :

Definition 12 (Point Guards). *These are guards that can be located anywhere in the polygon to be guarded.*

Definition 13 (Vertex Guards). *In this case, the positions of the guards are restricted to vertices of the polygon.*

The distinction between point and vertex guards is important. In many of our results, the bounds obtained for these two types of guards are different.

Definition 14 (Edge guards). *Edge guards were introduced by Toussaint in 1981. His original motivation was that of allowing a guard to move along the edges of a polygon. A point q can be considered guarded if it is visible from some point in the path of a guard. Alternately, we could think of the illumination problem of a polygon P in which we are allowed to place "fluorescent" lights along the edges of P ; each fluorescent light covers the whole length of an edge of P . Within this setting, our problem becomes: How many "fluorescent" lights are needed to illuminate a polygon with n vertices?*

Definition 15 (Mobile guards). *O'Rourke introduced this variation in which the guards are allowed to move along closed line segments totally contained in a polygon P .*

2.2.2 *Variations of Art Gallery Structure.*

Apart from variants arising from numerous placements for guards (G) and points to be covered (C), variants with different properties of polygon P have been vastly studied. One of the variations includes considering the input being an *Orthogonal Polygon*. This is denoted by appending an $' - O'$ to the problem notation ($XYG - O$). The input polygon P may even have some forbidden closed regions where guards are not allowed to be placed. Formally we define there forbidden closed regions as *Holes*. Given the polygon P and a set of m disjoint polygons $P_1, ..., P_m$ (called holes) contained in the interior of P, we call the set $P - \{P_1 \cup . . . \cup P_m\}$ a polygon with holes. We can also consider variant where holes are allowed inside polygons and denote them by $XYG(H)$ (with $'H'$ representing holes variant).

Table 1 summarizes all variants possible with symbols used in the paper.

Symbol	Definition
AGP	Art Gallery Problem
VG	Vertex Guard
EG	Edge Guard
PG	Point Guard
BG	Boundary Guard
VVG	Vertex-Vertex Guard
VBG	Vertex-Boundary Guard
BVG	Boundary-Vertex Guard
BBG	Boundary-Boundary Guard
PVG	Point-Vertex Guard
PBG	Point-Boundary Guard
EVG	Edge-Vertex Guard
EBG	Edge-Boundary Guard
VG(H)	Vertex Guard in Polygon with holes
EG(H)	Edge Guard in Polygon with holes
PG(H)	Point Guard in Polygon with holes
BG(H)	Boundary Guard in Polygon with holes
VVG(H)	Vertex-Vertex Guard in Polygon with holes
VBG(H)	Vertex-Boundary Guard in Polygon with holes
BVG(H)	Boundary-Vertex Guard in Polygon with holes
BBG(H)	Boundary-Boundary Guard in Polygon with holes
PVG(H)	Pont-Vertex Guard in Polygon with holes
PBG(H)	Point-Boundary Guard in Polygon with holes
EVG(H)	Edge-Vertex Guard in Polygon with holes
EBG(H)	Edge-Boundary Guard in Polygon with holes
VG − O	Vertex Guard on Orthogonal Polygon
VG(H) − O	Vertex Guard in Orthogonal Polygon with holes

Table 1. Glossary of all used notation.

3 HARDNESS OF AGP

Like a lot of other computational geometry problems, the question of tractability of AGP has been a long standing open problem. The problem has been proven to be NP-Hard as follows:

Theorem 1. *The minimum vertex guard problem (*VG*) for polygons is* NP*-Hard.*

*Proof Idea:*Their proof is based on a reduction of the 3-satisfiability problem [FK94]. In the same paper, they show that the minimum edge and point guard problems are also NP-hard. □

Similar NP-Hard results were established for vertex as well as point variants for orthogonal polygons in 1995 by Schuchardt et al.

Theorem 2. *[SH95] The minimum vertex-guard (*VG − O*) and point-guard (*PG − O*) problems for orthogonal polygons are* NP*-Hard.*

It has long been known that the problem is NP-hard, but no one has been able to show that it lies in NP. Recently, the computational geometry community became more aware of the complexity class $\exists R$, which has been studied earlier by other communities. The class $\exists R$ consists of problems that can be reduced in polynomial time to the problem of deciding whether a system of polynomial equations with integer coefficients and any number of real variables has a solution. Clearly $NP \subseteq \exists R$.

Working in this direction, in 2018, Abrahamsen et al. proved that solving the AGP is as hard as deciding whether a system of polynomial equations and inequalities over the real numbers has a solution under a polynomial time reduction. This implicitly would imply that (1) any system of polynomial equations over the real numbers can be encoded as an instance of the AGP, and (2) the AGP is not in the complexity class NP unless $NP = \exists R$. The main contribution of their work is following theorem:

Theorem 3. *[AAM18, Theorem 1.1] Point Guard (*PG*) variant of* AGP *is $\exists R - complete$, even the restricted variant where we are given a polygon with corners at integer coordinates.*

With a few modifications it is possible to extend this result to Edge Guard (EG) variant as well.

3.1 Irrational Placement of Guards

In 2011, Dagstuhl [AMT11] pitched an open problem asking whether there exist polygons such that every optimal solution requires at least one guard to be placed at irrational coordinates. Despite a tremendous amount of work on the AGP, the open problem was not solved until 2017 by Abrahamsen et al. [AAM17] who gave a positive existential results by constructing a family of monotone polygons given by integer coordinates that require guard positions with irrational coordinates in any optimal solution. They proved that ratio of optimal number of guards with only rational coordinates to optimal number of guards with irrational coordinates for this family of polygons is $\frac{4}{3}$.

The building block of proof involves constructing a polygon (as shown in Figure 1) which requires 3 guards if we are allowed to place guards at irrational coordinates. Otherwise, four guards are needed if only rational coordinates are allowed. The proof is further extended such that for every natural number n, we have a monotone polygon which requires $3n$ guards with irrational coordinates. Otherwise, $4n$ guards are necessary if only rational coordinates for guards are allowed. Their results are summarized as follows:

Theorem 4. *[AAM17, Theorem 1] There exists a simple monotone polygon P (as shown in Figure 1) such that the coordinates of vertices are integral values and P can be optimally guarded by* 3 *guards with irrational coordinates whereas with only rational coordinates allowed for placing the guards* 4 *guards are required optimally for guarding P.*

Theorem 5. *[AAM17, Theorem 1] There exists a family of simple polygons $(P_n)_{n\in Z_+}$ (as shown in Figure 2) such that the coordinates of vertices are integral values and each P_n can be optimally guarded by* $3n$ *guards with irrational coordinates whereas with only rational coordinates allowed for placing the guards* $4n$ *guards are required optimally for guarding P_n.*

Another extension to these results proves that a similar ratio between irrational optimal solution and rational optimal solution is observed in the class of rectilinear polygons.

As an open problem Friedrichs et al. [FHKS15] questioned : "For the Art Gallery Problem it is not known whether the coordinates of an optimal guard cover can be represented with a polynomial number of bits". It remained as an open problem, till 2018. In the paper by Abrahamsen et al. [AAM18] they prove that under the assumption $NP \neq \exists R$ the AGP is not in NP, and such a representation does not exist.

Another result, which follows as a corollary of the construction for hardness proof in Theorem 3, proves that for any real algebraic number α, there is an instance of the AGP where one of the coordinates of the guards equals α in any guard set of minimum cardinality. This is a generalization of Theorem 5, which established that irrational guards are sometimes needed in optimal guard sets.

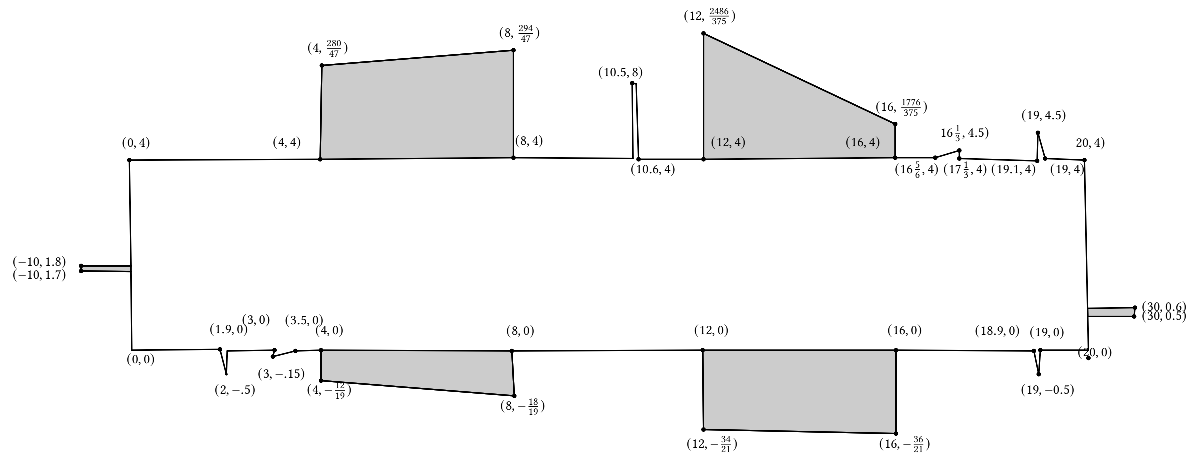


Fig. 1. Polynomial with 3 irrational guards in optimal solution but requires 4 rational guards (not to scale).

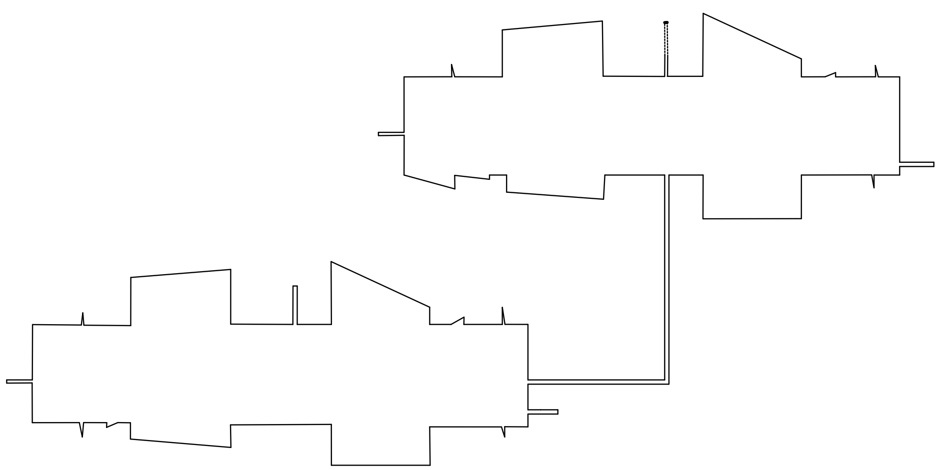


Fig. 2. Polynomial with 6 irrational guards in optimal solution but requires 8 rational guards.

Theorem 6. *[AAM18, Theorem 1.2] Given any real algebraic number α, there exists a polygon P with corners at rational coordinates such that in any optimal guard set of P there is a guard with an x-coordinate equal to α.*

Theorem 6 justifies the difficulty in constructing algorithms for the AGP, explains the lack of combinatorial algorithms for the problem and rules out many algorithmic approaches to solving the AGP.

Another major result in [AAM18] proved that every instance of the AGP can be encoded as an existential formula using $(n + k)^6$ variables, where k is the number of guards and n is the number of corners of the polygon. This brings up a new problem of optimizing the number of variables that are sufficient as well as necessary for encoding the problem, as fewer variables lead to faster algorithms for the AGP.

4 BOUNDS ON NUMBER OF GUARDS

In this section we deviate from the goal of computing an optimal solution for AGP and present works done in the direction of proving various bounds on the number of guards required (or sufficient). The section has been split into two sections, Section 4.1 elaborates results for the classical variant on simple polygons, whereas in Section 4.2 we present results for AGP under the assumption that input polygon is orthogonal.

4.1 Classical Art Gallery

Throughout the paper, we refer to a classical art gallery as an art gallery whose flooring is a simple polygon. In this section, we discuss important results related to classical art gallery, with different kind of guards. The first problem related to AGP was posed by Victor Klee in 1973, where he asked the minimum number of stationary guards to guard an art gallery. The first result, as proved by Vaclav Chvàtal in 1975 is :

Theorem 7. $\lfloor \frac{n}{3} \rfloor$ *stationary guards are always sufficient and occasionally necessary to illuminate a polygonal art gallery with n vertices.*

Proof Idea: He proves that every n-triangulation can be partitioned into m fans where $m \leqslant \lfloor \frac{n}{3} \rfloor$ and the proof is based on induction. For base case, for $n = 3, 4, 5$ the result is trivial as any n-triangulation itself would be a fan triangulation. For the inductive case, for the n-triangulation (say G), mark the vertices $1, 2, ..., n$ in cyclic order. Let k denote the smallest integer such that $k \geqslant 4$ and G has an edge $(j, j+k)$ for some j. With this definition, we have $k \leqslant 6$. Now, notice that this edge $(j, j+k)$ divides G into a $(k+1)$-triangulation G_1 and $(n-k+1)$-triangulation G_2. They then consider different cases which arise for different values of k and show that the triangulation G can be partitioned into m fans with $m \leqslant \lfloor \frac{n}{3} \rfloor$. This gives us the required result, as after triangulation of the art gallery, we can partition it into m fans with $m \leqslant \lfloor \frac{n}{3} \rfloor$. As each fan can be guarded with a single guard, this would guard the entire art gallery. □

The proof by Chvàtal was long and tedious. Fisk provided a much simpler proof for the same using vertex coloring.
Proof Idea: For the simple polygon, we perform a triangulation which can be done by adding $O(n)$ interior diagonals. (See Figure 3(a)) . Consider the graph with vertices as vertices of the polygon and edges as the edges from this triangulation. We can perform a 3-coloring of this graph always (despite the problem of 3-coloring of a graph being NP-complete). Once a 3-coloring is performed, we pick the vertices from the color class of smallest size. Say this size is m. We can gurantee that $m \leqslant \lfloor \frac{n}{3} \rfloor$, as m is the minimum of size of the color classes, and $\frac{n}{3}$ is the average of the size of the three classes, and we know that minimum of a set of numbers is less than or equal to their average. If we place guards at the vertices colored by this minimum class-size color, then it guards the whole art gallery. Because, for each triangle of the triangulation, we have one guard at one of its vertices (as we have performed a 3-coloring), and this guard would cover the triangle. Hence, all triangles are guarded, which means the whole art gallery will be guarded. □

These results show that $\lfloor \frac{n}{3} \rfloor$ are sufficient. To show that they are sometimes necessary, consider the family of polygons known as comb polygon. For example, consider the comb polygon C_m with $n = 3m$ vertices shown in Figure 3(b). It is not difficult to observe that at least m guards will be required to cover a comb polygon. Hence, these families show that sometimes $\lfloor \frac{n}{3} \rfloor$ are required.

4.1.1 Holes In Classical Art Gallery.

Here, we will discuss the problem variant where holes can be present in the art gallery. These holes complicate the problem further as they obstruct the vision of the guards, mostly leading to increased number of required guards in the optimal solution. A fundamental lemma, as shown by Rourke [O'r87] says :

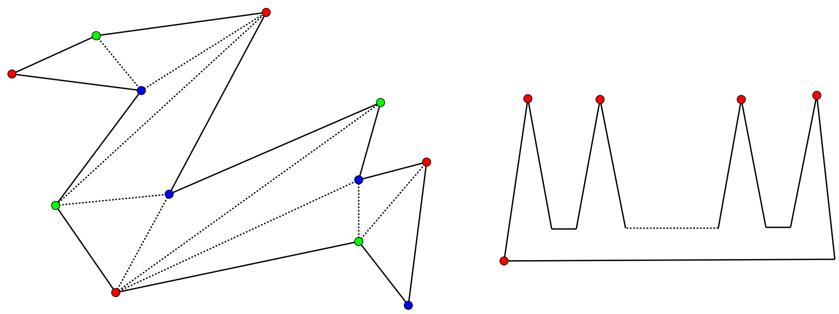


Fig. 3. Illustration of proof of Chvàtal's Art Gallery Theorem. (a) A triagulated polygon, with 3-colored vertices (green, blue, red). (b) Comb Polygon : Tight case for Theorem 7 with $\frac{n}{3}$ guards (One possible guard placement is shown with red nodes).

Lemma 1. *Triangulation of a polygon with holes is always possible*

Proof Idea: Consider that polygon P has n vertices and h holes. The proof is based on induction, with base case of $h = 0$ which is known to be true. Consider a completely internal diagonal d of this polygon. Imagine splitting the polygon along this diagonal. Two cases arise. First case is where one end point of this diagonal lies on a hole. After split, a single polygon remains, but h decreases by 1 and n increases by 2 (The hole sharing vertex with diagonal disappears). In the second case, both end points lie on the outer boundary, then the split results in two polygons, both having lesser number of vertices and holes. In both cases, the inductive hypothesis holds and result follows. □

O'Rourke [O'r87] proved the first result on guarding polygons with holes:

Theorem 8. *[O'r87] Any polygon with n vertices and h holes can always be guarded with* $\lfloor \frac{(n+2h)}{3} \rfloor$ *vertex guards.*

Proof Idea: The proof eliminates the holes of polygon repeatedly. For the given polygon P, suppose it has been triangulated into a triangulation T (which we know can be done from Lemma 1). Now, for every hole, we would have some diagonal with one of its endpoint on it. The other end may be on another hole, or to a vertex on the outer boundary of the polygon. If a split is performed along this diagonal, then in the former case, the two holes will merge into a single hole, and in the latter case, the hole and outer region get merged effectively removing the hole. In both cases, the number of holes reduce by 1. However, not all sequences of cuts result in a single polygon at the end. A method of choosing cuts which uses dual of the triangulation was mentioned by Rourke which guarantees a single polygon at the end. The important part for the proof however, is that after all holes are cut, the resulting polygon P' would have $n + 2h$ vertices. This is because each cut would produce two new vertices. As we now have a single polygon without holes, applying the classical art gallery theorem gives the desired result of $\lfloor \frac{(n+2h)}{3} \rfloor$ vertex guards. □

This bound is not believed to be tight. Shermer showed examples of polygons with one hole which can easily be stitched together for larger number of vertices and holes as shown in Figure 5. Using this, he showed that :

Theorem 9. *[O'r87] For polygon with n vertices and h holes,* $\lfloor \frac{(n+h)}{3} \rfloor$ *vertex guards are sometimes necessary.*

He further conjectured that :

Conjecture 1. *Any polygon with n vertices and h holes can always be guarded with* $\lfloor \frac{(n+h)}{3} \rfloor$ *vertex guards.*

This conjecture was proved by Shermer for $h = 1$[O'r87], but remains open for $h > 1$.

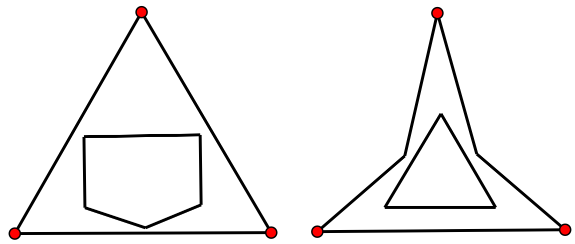


Fig. 4. Polygons with n=8, h=1 and require 3 guards (One possible guard placement is shown with red nodes).

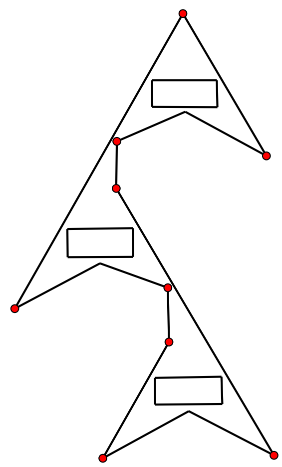


Fig. 5. A stitched polygon with 24 vertices, 3 holes and requires 9 guards (One possible guard placement is shown with red nodes).

We will now discuss some results for classical art gallery with holes and using point guards, where we have an interesting result which looks similar to Shermer's Conjecture. Bjorling-Sachs and Souvaine [BSS91] and Hoffmann, Kaufman, and Kriegel [HKK91] independently proved:

Theorem 10. *[BSS91] [HKK91]* $\lfloor \frac{(n+h)}{3} \rfloor$ *point guards are always sufficient and occasionally necessary to guard any polygon with n vertices and h holes.*

Proof Idea: The approach we briefly mention here is the one used by Bjorling-Sachs and Souvaine[BSS91]. They first connect each hole to the exterior with a quadrilateral "channel" making the polygon hole-free as holes have been merged with the exterior. The channels are such that only one new vertex is introduced for each channel, and a triangle T in the remaining polygon sees all of the channel. Then they triangulate the hole-free version of the polygon,while ensuring that this triangle T exists in the triangulation and place guards based on three-coloring. This gives the desired bound of $\lfloor \frac{(n+h)}{3} \rfloor$. □

In 1995, Bjorling-Sachs and Souvaine [BSS] gave an $O(n^2)$ time algorithm to find the position of the $\lfloor \frac{(n+h)}{3} \rfloor$ point guards. Their proof and algorithm does not apply to vertex guards, as they modify the polygon during which several new vertices were created, which may not necessarily exist in the original polygon and therefore would not correspond to vertex guards. For vertex guards, the problem of finding a tight bound remains open.

4.1.2 Edge and Mobile Guards in Classical Art Gallery.

The problem of edge guards was proposed in 1981, where Toussaint asked the question of determining the minimum number of edge guards required to guard any polygon with n vertices. He made the following conjecture:

Conjecture 2. *Except for a few polygons (as shown in 6(b)), $\lfloor \frac{n}{4} \rfloor$ edge guards are always sufficient to guard any polygon with n vertices.*

In Figure 6 in the left we have an example polygon that requires $\lfloor \frac{n}{4} \rfloor$ edge guards, and in the right, we have the only two known counterexamples to this conjecture which were given by Paige and Shermer.

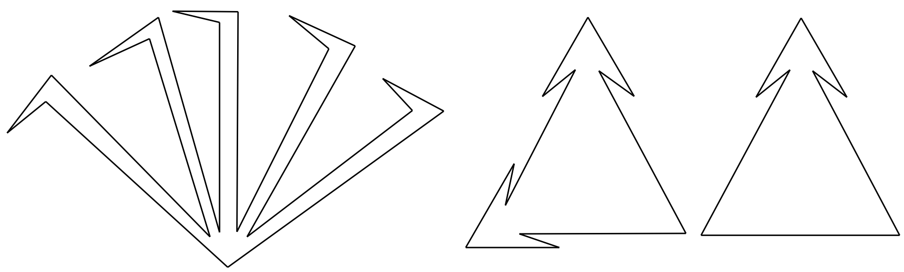


Fig. 6. (a) A polygon that requires $\lfloor \frac{n}{4} \rfloor$ edge guards (b) Paige and Shermer's polygons requiring more than $\lfloor \frac{n}{4} \rfloor$ edge guards

The first positive result in this direction was given by O'Rourke [O'R83], which was for mobile guards. By allowing the guards to move along diagonals joining vertices of P, i.e. using mobile guards, he proved the following:

Theorem 11. *[O'R83] $\lfloor \frac{n}{4} \rfloor$ mobile guards are always sufficient and occasionally necessary to guard any polygon with n vertices.*

Proof Idea: O'Rourke's proof is based on induction on the number of vertices of P_n. He first establishes the result for polygons with up to nine vertices as base case. As the inductive step, for polygons with more than nine vertices, he considers a triangulation T of P_n. Then he shows that this triangulation contains a diagonal that cuts P_n into two polygons, P' and P'', one of which, say P', contains between five and eight edges of P. He then finds a solution for P' that can be used with any solution of P'' to obtain a solution for P_n. It was also shown that O'Rourke's proof of Theorem 11 can be implemented in linear time [O'R83]. □

An interesting problem arose at this stage. As defined before a triangulation graph is a maximal outer planar graph, i.e. a Hamiltonian planar graph which contains n vertices and $2n$-3 edges, and all of whose internal faces are triangles. O'Rourke showed that there are triangulation graphs with n vertices such that any set of edges that covers their triangular faces requires $\lfloor \frac{2n}{7} \rfloor$ edges. Shermer later found examples of triangulation graphs that require $\lfloor \frac{3n}{10} \rfloor$ edge guards to cover them. Since this number is greater than $\lfloor \frac{n}{4} \rfloor$, this means that the technique of trying to solve Toussaint's conjecture using triangulation may not work.

Shermer [She94] showed the following important result for edge guarding of triangulated graphs:

Theorem 12. *[She94] $\lfloor \frac{3n}{10} \rfloor$ edge guards are always sufficient and occasionally necessary to guard any triangulation graph with n vertices, with the exception of three graphs.*

Proof Reference. The proof of Theorem 12 is complicated and very long, and hence is omitted from here. Please refer to Shermer's original paper for more details. □

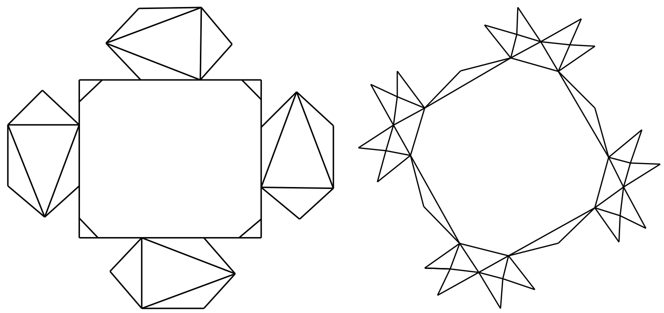


Fig. 7. Tight Cases $\lfloor \frac{2n}{7} \rfloor$ and $\lfloor \frac{3n}{10} \rfloor$ edge guards respectively.

For spiral polygons, I. Bjorling-Sachs [BS94] and S. Viswanathan [Vis93] showed that $\lfloor \frac{n+2}{5} \rfloor$ edge guards are always suffcient and sometimes necessary. For finding these guards, linear time algorithms are also provided in [BS94].

Before concluding the results on classical art gallery problem, we would like to briefly mention a connection it has with Fary's Theorem.

Theorem 13 (Fary's Theorem). *Any simple planar graph can be drawn on a plane without crossings with every edge as a straight-line segment.*

We have defined planar graphs in Definition 8. But in general there is no requirement for the edges to be straight i.e. they can be arbitrarily shaped lines. Hence, this theorem becomes important for many applications. *Proof Idea:* For simplicity in the idea, we consider only maximally planar graphs i.e. planar graphs where addition of any new edge makes it non-planar. From simple observation, we can claim that these maximally planar graphs have all faces as triangles. If not, we could have added any edge inside the face and still have a planar graph, thus contradicting the definition of maximal planar graph. The proof idea begins with fixing one the faces as a triangle to act as the outer triangular face, and then shows that the rest of the graph can be drawn with straight lines using induction. Suppose we have shown the result for planar graph with n vertices. Now, for the graph with $n+1$ vertices, we know that there will be a vertex with degree $\leqslant 5$ in the graph, and can assume it is not one of the vertices in the outer triangular face for simplicity. On removing this vertex (along with incident edges), we get a graph with n vertices. To keep it maximal planar, we triangulate this face. Now, by induction, we can draw this graph in plane with straight edges. Now we can remove the extra edges which we added to make the n vertex graph maximal planar. The only part that remains is to re-add the vertex and its incident edges (as straight lines) back into this new embedding. Here, we can use the results from art gallery. The face in question will have at most five sides, and we need at most $\lfloor \frac{5}{3} \rfloor = 1$ guards to cover the gallery. If it covers the gallery, it also sees all the vertices of the face (i.e. all vertices are visible to it). On placing the $(n+1)th$ vertex here, we can therefore connect it to all of the other vertices with straight lines. This gives the desired embedding of the graph with $n+1$ vertices. □

4.2 Orthogonal Art Gallery

With the original problem of Chvátal solved, interest turned to different variations of the AGP. One such version is when instead of a general polygon, the gallery is assumed to be bounded by an orthogonal polygon. In 1980, Kahn, Klawe, and Kleitman proved that :

Theorem 14 (Kahn, Klawe and Kleitman). *[KKK83] For any orthogonal polygon with n vertices, $\lfloor \frac{n}{4} \rfloor$ vertex guards are always sufficient to guard it. There exist polygons (as shown in 8), where $\lfloor \frac{n}{4} \rfloor$ vertex guards are necessary.*

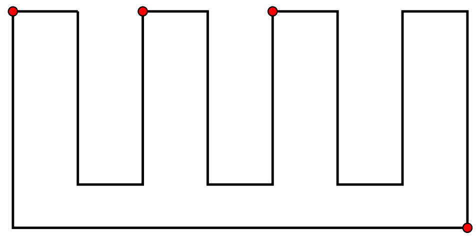


Fig. 8. Tight case for Theorem 14 where $\lfloor \frac{n}{4} \rfloor$ guards always required. Here, $n = 16, g = 4$ (shown with red nodes).

Proof Idea: The proof by Kahn, Klawe and Kleitman uses the same technique which was used by Fisk for proving the $\lfloor \frac{n}{3} \rfloor$ bound for classical art gallery. For orthogonal art gallery, the main idea of the proof is to partition the orthogonal polygon into convex quadrilaterals. Then, internal diagonals of each of these quadrilaterals are added, and the obtained graph is four-vertex colored. Finally, we pick the minimum size color class and place guards there, which is a valid positioning as it covers the entire art gallery because for each quadrilateral, we have a guard at one of its vertices. A guard at the vertex of the convex quadrilateral covers the quadrilateral, and we have one guard at each quadrilateral, hence the whole gallery is covered. The bound here is $\lfloor \frac{n}{4} \rfloor$ as average of the sizes of color classes will be $\frac{n}{4}$, and minimum is smaller than or equal to average. An interesting fact is that Kahn, Klawe and Kleitman's result provided an incentive to study the problem of decomposing a rectilinear polygon into convex quadrilaterals. The first $O(n)$ algorithm to achieve this was obtained by Sack [SoCS84]. □

Győri and O'Rourke presented clear and brief confirmation of Theorem 14 separately in the first half of the 1980s.

Theorem 15. *[Gyö86][O'r87, Theorem 2.5] It is possible to divide of orthogonal polygon of n vertices into $\lfloor \frac{n}{4} \rfloor$ orthogonal polygons of at most 6 vertices.*

In certain ways, Theorem 15 is a deeper outcome than that of Kahn, Klawe, and Kleitman, as a stationary guard will protect any plain orthogonal polygon of 6 vertices. So far, each evidence sheds light on a fascinating concept that we shall refer to as the "metatheorem":

Metatheorem 1. *[Mez17, Metatheorem of art galleries] Each theorem of the (orthogonal) art gallery has an underlying partition theorem (in simple terms).*

The metatheorem is also verified by proof of the sharp bound on mobile guards in simple polygons according to Theorem 9. Although both Theorem 14 and Theorem 15 are only related to art galleries, Hoffman showed that for any closed area bounded by parallel line segments of the axes, the same bound applies.

Theorem 16. *[Hof90] Any orthogonal polygon with holes of a total of n vertices can be partitioned into $\lfloor \frac{n}{4} \rfloor$ rectangular stars of at most 16 vertices.*

The Metatheorem 1 is also confirmed by Theorem 16. Hoffmann and Kaufmann provided an appropriate algorithm to create such a partition soon after this result. We offer further proof that the metatheorem holds, namely that the following partition theorem is proved:

Theorem 17 (Győri and Mezei). *Any simple orthogonal polygon of n vertices can be partitioned into at most $\lfloor \frac{3n+4}{16} \rfloor$ orthogonal polygons of at most 8 vertices.*

The follwoing bound holds w.r.t the number of reflex vertices in the art gallery:

Theorem 18. *[CHH] For an orthogonal polygon* $\lfloor r/2 \rfloor + 1$ *guards are always sufficient and occasionally necessary to guard the whole polygon, where r is the number reflex vertices.*

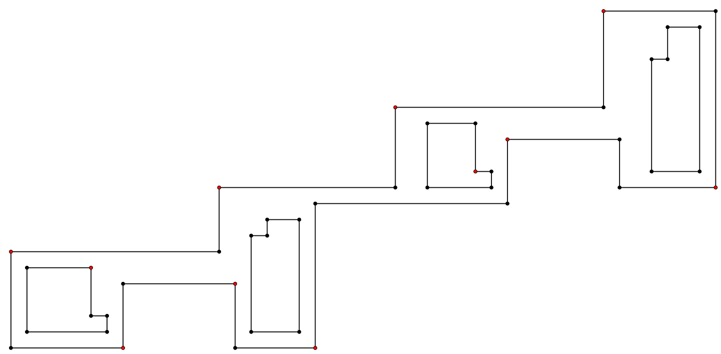


Fig. 9. An orthogonal polygon with 44 vertices and 4 holes that requires 12 vertex guards.

4.2.1 Holes in Orthogonal Art Gallery.

In 1982, O'Rourke [O'r87] proved that any orthogonal polygon with n vertices and h holes can always be guarded with$\lfloor \frac{n+2h}{4} \rfloor$ vertex guards. His proof was similar to the proof for classical art gallery with holes. Further, he conjectured that $\lfloor \frac{n}{4} \rfloor$ point guards are always sufficient to guard the art gallery. Aggarwal [Agg84] was able to verify this conjecture for number of holes $h = 1, 2$. It remained as an open conjecture until in 1990, F. Hoffmann [Hof90] proved:

Theorem 19. *[Hof90]* $\lfloor \frac{n}{4} \rfloor$ *point guards are always sufficient to guard any orthogonal art gallery with n vertices and h holes.*

The above theorem was for point guards. For vertex guards, the best known upper bound, as mentioned above about O'Rourke's bound, remains at $\lfloor \frac{n+2h}{4} \rfloor$. Many situations are known where $\lfloor \frac{n}{4} \rfloor$ vertex guards are not sufficient to guard orthogonal polygons with many holes. For example, the polygon shown in Figure 9 , with 44 vertices and 4 holes, requires 12 vertex guards. This example, which can easily be generalized, inspired T. Shermer [O'r87] to make the following conjecture:

Conjecture 3. *[O'r87]* $\lfloor \frac{n+h}{4} \rfloor$ *vertex guards are sufficient to cover any orthogonal polygon with n vertices and h holes.*

For the case when h is large enough $h \geqslant \frac{n}{6} - 2$, O'Rourke's upper bound of $\lfloor \frac{n+2h}{4} \rfloor$ on the number of vertex guards needed to guard an orthogonal art gallery with holes was achieved by Hoffman and Kriegel [HK93]. They proved:

Theorem 20. *[HK93]* $\lfloor \frac{n}{3} \rfloor$ *vertex guards are always sufficient to guard an orthogonal polygon with holes.*

Proof Idea: Their proof requires knowing the following result about planar bipartite graphs :

Lemma 2. *Any planar triangulated graph is 3-vertex colorable iff all its vertices have even degree.*

To prove their result, they first partition the orthogonal polygon into convex quadrilaterals, and then triangulate the resulting graph in such a way that every vertex has even degree. Then they use Lemma 2 and infer that any planar triangulated graph is 3-vertex colorable iff all its vertices have even degree. Their result leads to an $O(n^2)$ time algorithm. □

All of the results we have seen propose an upper bound on the sufficient number of vertex guards. However, how much tighter can it be made? For the lower bound on these sufficiencies, Hoffman showed a family of orthogonal polygons with holes which require $\lfloor \frac{2n}{7} \rfloor$ vertex guards as shown in figure using the tight example Figure 10. This disproved an earlier conjecture by Aggarwal [Agg84] that $\lfloor \frac{3n}{11} \rfloor$ were always sufficient.These family of polygons also suggest that we cannot hope to get a result which proposes upper bound smaller than $\lfloor \frac{2n}{7} \rfloor$. Following this, Hoffman made the following conjecture:

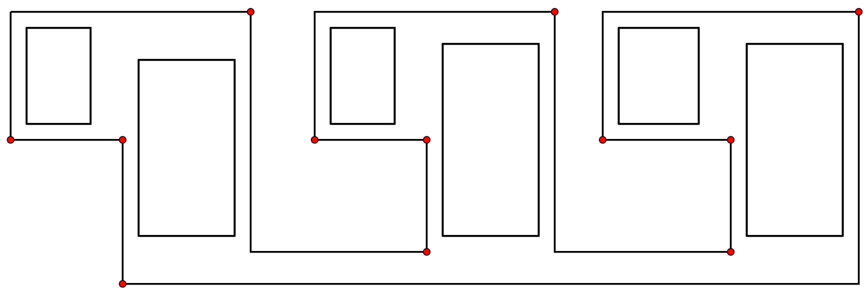


Fig. 10. Hoffman's family of polygons requiring $\frac{2n}{7}$ guards (One possible placement is shown with red nodes)

Conjecture 4. $\lfloor \frac{2n}{7} \rfloor$ *vertex guards are always sufficient to guard any orthogonal polygon with holes.*

Remember that Shermer's conjecture was for the bound of $\lfloor \frac{n+h}{4} \rfloor$, and Hoffman's polygons are not disproving Shermer's conjecture that $\lfloor \frac{n+h}{4} \rfloor$ vertex guards are sufficient to cover any orthogonal polygon. Interestingly, Hoffman's polygons have $n = 14k$ vertices and $h = 2k$ holes, and for this particular choice of numbers, $\lfloor \frac{2n}{7} \rfloor = \lfloor \frac{n+h}{4} \rfloor$.

4.2.2 Edge and Mobile Guards in Orthogonal Art Gallery.

A mobile guard, as defined in 15 is one that is willing to patrol a line segment of the museum. The upper bound of the theorem of the mobile guard art gallery for orthogonal polygons follows immediately from Theorem 17, as a mobile guard can cover an orthogonal polygon of at most 8 vertices.

Another result for mobile guards in orthogonal polygons was shown by A. Aggarwal [Agg84], as follows:

Theorem 21. *[Agg84]* $\lfloor \frac{3n+4}{16} \rfloor$ *mobile guards are sufficient and occasionally necessary as shown in Fig Figure 11 to cover any orthogonal polygon with n vertices.*

Proof Idea: The proof of Aggarwal's result is interesting, but rather long and complicated, and hence details are omitted from here. The complete proof can be found in [O'r87]. Interestingly, Aggarwal's proof for Theorem 21 leads to an $O(n^2 logn)$ time algorithm using Sack's or Lubiw's quadrilaterization algorithms. □

The lower bound for Theorem 21 and Theorem 17 is given by a sequence of swatikas (Figure 12) stringing together. Note that, for a limit of one arm, a mobile guard can cover certain points of the end of a swastika arm. Therefore, any arm must be equipped with a mobile guard. A spiral needs to be attached to one of the arms for $n = 0(mod16)$.

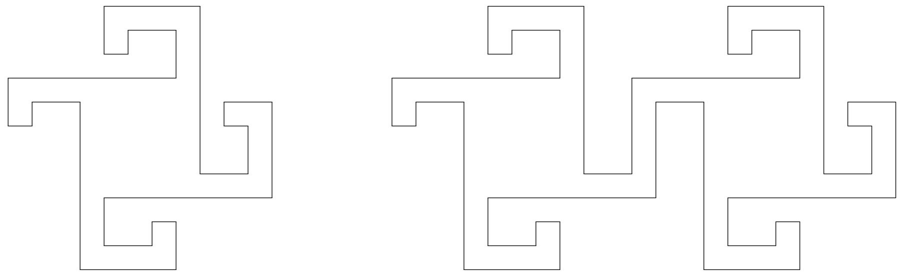


Fig. 11. Tight Cases $\lfloor \frac{3n+4}{16} \rfloor$ mobile guards.

A better finding than Theorem 17 is Theorem 21, and it is interesting on its own. It falls into the set of findings in [[Gyö86];[O'r87, Theorem 2.5]; [GHKS96]] showing that theorems of orthogonal art gallery are focused on partition theorems in smaller parts ('one guardable').

In addition, the following corollary that confirms the previous theorem and addresses two questions posed by O'Rourke [O'r87, Theorem 3.4] explicitly implies Theorem 17.

Corollary 1 (Győri and Mezei). $\lfloor \frac{3n+4}{16} \rfloor$ *mobile guards are adequate to cover a basic orthogonal polygon of n-vertex such that the two guards' patrols do not pass through each other and only at the endpoints of the patrols is visibility required.*

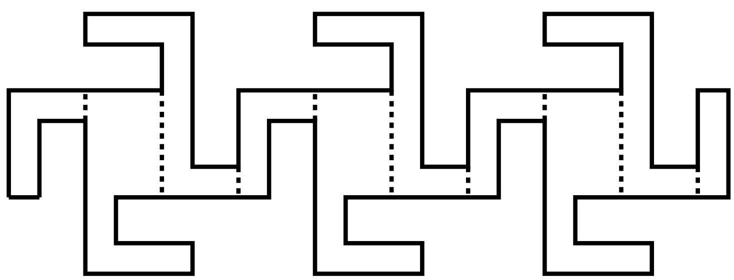


Fig. 12. The dashed lines show a minimum cardinality partition into at most 8-vertex pieces.Not to scale

In his 1987 book entitled 'Art Gallery Theorems and Algorithms' [O'r87], Joseph O'Rourke pointed out that there is a curious 4 : 3 ratio between the extreme number of points and mobile guards for art galleries provided by both simple polygons and simple orthogonal polygons.

Aggarwal's result was generalized to orthogonal polygons with holes by Gyory, Hoffmann, Kriegel and Shermer [GHKS96]. They proved:

Theorem 22. *[GHKS96]* $\lfloor \frac{3n+4h+4}{16} \rfloor$ *mobile guards are always sufficient and occasionally necessary to guard an orthogonal polygon.*

For spiral orthogonal polygons, I. Bjorling-Sachs [BS94] proved that $\lfloor \frac{n-2}{6} \rfloor$ edge guards are always sufficient, and occasionally necessary. There's also a linear time algorithm provided to find these guards.

5 APPROXIMATION ALGORITHMS

We begin with defining approximation algorithms and approximation guarantee.

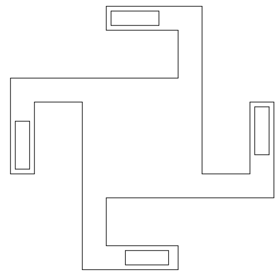


Fig. 13. Tight Cases $\lfloor \frac{3n+4h+4}{16} \rfloor$ mobile guards.

Definition 16 (Approximation Guarantee). *For a minimization problem Π, an algorithm $\mathcal{A}$ has approximation guarantee of α ($\alpha > 1$), if $\mathcal{A}(I) \leqslant \alpha\ OPT(I)$ for all input instance I of Π. For a maximization problem Π', an algorithm $\mathcal{A}$ has approximation guarantee of α ($\alpha > 1$), if $OPT(I) \leqslant \alpha\ \mathcal{A}(I)$ for all input instance I of Π'.*

Definition 17 (Polynomial Time Approximation Scheme (PTAS)). *A minimization problem Π admits PTAS if for every constant $\varepsilon > 0$, there exists a $(1+\varepsilon)$-approximation algorithm with running time $O(n^{f(1/\varepsilon)})$, for any function f that depends only on ε.*

The class *APX* (an abbreviation of "approximable") is the set of NP optimization problems that allow polynomial-time approximation algorithms with approximation ratio bounded by a constant (or constant-factor approximation algorithms for short).

Definition 18 (APX-Harndess). *A problem is said to be APX-hard if there is a PTAS reduction from every problem in APX to that problem.*

If the running time of a PTAS is $O(f(1/\varepsilon)\ n^c)$ for some function f and a constant c that is independent of $1/\varepsilon$, we call it Efficient PTAS (EPTAS), If the running time of a PTAS is polynomial in both n and $1/\varepsilon$, we call it Fully PTAS (FPTAS). Quasi-polynomial time approximation scheme (QPTAS) and pseudo-polynomial time approximation scheme (PPTAS) are defined analogously as PTAS, however, their running times are quasi-polynomial (i.e., $n^{(\log n)^c}$ for some constant $c > 1$) and pseudo-polynomial time, respectively. Asymptotic analogue of PTAS, EPTAS, FPTAS are known as APTAS, AEPTAS, AFPTAS, respectively. We refer the reader to [CKPT17] for more on these approximation schemes and their connections with hardness assumptions.

The section is divided in 4 subsections. In Section 5.1, we establish inapproximability results for various versions of AGP. In Section 5.2, we state the $O(logn)$-approximation result by Ghosh, realised by a reduction to Set Cover Problem. Further, we state $O(logOPT)$ result by Miltzow et al. in Section 5.3. We then deviate to variant of the problem, with weak visibility polygons, and state the result of 6-approximation by Ghost et al. [Section 5.4]. In the end, in Section 5.5 we quote some more results for AGP.

5.1 Inapproximability Of AGP

In 2001, Eidenbenz et al. [ESW01] established inapproximability results for most of the variants of AGP proving that there is no c-approximation algorithm for simple polygons; for some constant c; exists for vertex, edge or point guard variants. The inapproximability holds even if only bounday has to be gaurded, i.e. for Vertex-Boundary (VBG), Point-Boundary (PBG) and Edge-Boundary (EBG) guard variants. The inapproximability results for simple polygons without holes can be summarized as follows:

Theorem 23. *[ESW01] Vertex, Point and Edge guards variants of* AGP *(*VG, PG, EG*) in simple polygons are APX − hard. Also Vertex-Boundary (*VBG*), Point-Boundary (*PBG*) and Edge-Boundary (*EBG*) guard variants of* AGP *in simple polygons are APX − hard.*

Proof Idea: The proof is done using gap-preserving reductions from $5 - OCCURRENCE - MAX - 3 - SAT$. □

Eidenbenz et al. [ESW01] futher extended the inpproximability results for polygons with holes proving that there exists no $o(logn)$-approximation algorithm running in polynomial time, unless $NP \subseteq TIME(nO(loglogn))$. We can summarize the inapproximatibility results for polygon with holes as follows:

Theorem 24. *[ESW01] There exists no polynomial time algorithm for Vertex Guard (*VG*), Point Guard (*PG*) or Edge Guard (*EG*)* AGP *with an approximation ratio better than* $(\frac{(1-\varepsilon)}{12})(lnn)$ *for every* $\varepsilon > 0$*, unless* $NP \subseteq TIME(nO(loglogn))$*, where n is the number of vertices of input polygon.*

5.2 $O(logn)$−Approximation [Gho10]

A simple polygon is called a fan if there exists a vertex that is visible from all points in the interior of the polygon. Intuitively the algorithm builds up as follows: First discretize the entire region of a polygon and then use the minimum set-cover solution. The vertex guard problem can be treated as a polygon decomposition problem in which the decomposition pieces are fans. The polygonal region of P is decomposed into convex components where every component is bounded by segments that contains any two vertices of the polygon. Every convex component must lie in at least one of the fans chosen by the approximation algorithm.

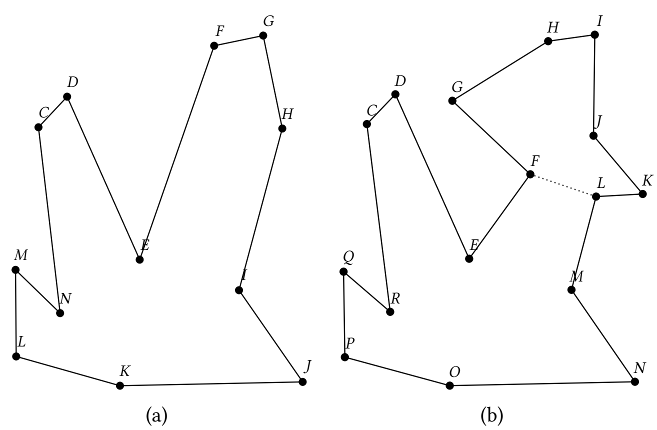


Fig. 14. (A) A simple polygon. (B) Dotted line separating upper fan polygon.

We now begin with the formal framework used in the algorithm. Consider a simple polygon P with its vertices labelled as $v_1, v_2, ...v_n$ in clockwise order. Let the boundary of polygon be $bd(P)$, and $bd_c(p, q)$ be the clockwise boundary from p to q. Similarly, we define $bd_{cc}(p, q)$ be the counter-clockwise boundary from p to q. We can observe that these definitions give us $b_c(p, q) = b_{cc}(q, p)$, and $bd(P) = b_c(p, p) = b_{cc}(p, p)$. For any point z, we define a visibility polygon $VP(z)$ as the set of all points in P that are visible from z. Mathematically, $VP(z) = \{q \in P : q \text{ is visible from } z\}$. The edges of this visibility polygon may not necessarily be all on the boundary of the original polygon. These non-polygonal edges will be called as constructed edges. Now, notice that for these constructed edges, one of the endpoints will be a vertex of P (call it v_i), while the other will lie

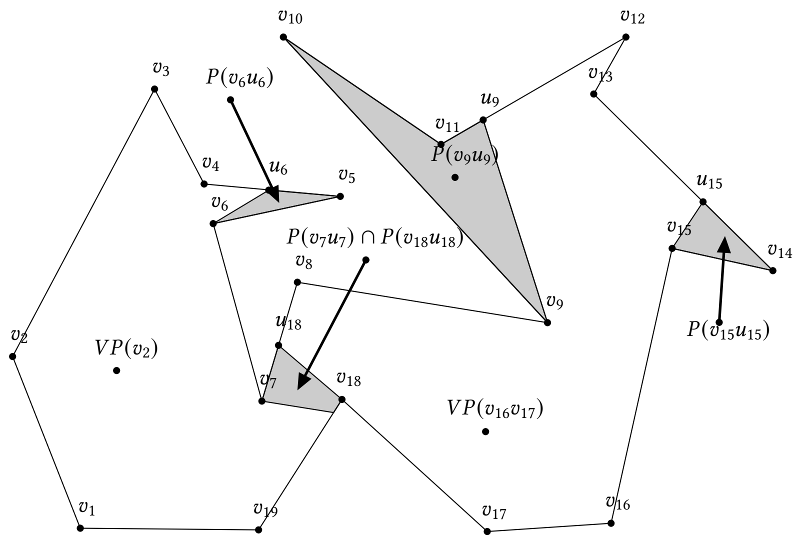


Fig. 15. Visibility polygon $VP(v_2)$; Weak visibility polygon $VP(v_{16}v_{17})$; some shaded pockets.

on the $bd(P)$ (call it u_i). These points z, v_i, and u_i will also be collinear. An example can be seen in Figure 15, consider $VP(v_2)$ and the constructed edges v_6u_6 and v_7u_7. We will now define weak visibility. Consider an edge or an internal chord bc. We call a point q of P as weakly visible from bc if there is a point on bc from which q is visible. In other words, q is weakly visible from bc if there exists a point z on bc such that q is visible from z. We can now define weak visibility polygon of P from bc $VP(bc)$ as set of all points of P which are weakly visible from bc. Further, if there is an edge of Pv_iv_{i+1} such that $VP(v_iv_{i+1}) = P$, then P is called a weak visibility polygon. $VP(bc)$ has both polygonal edges and constructed edges defined similar to $VP(z)$. We call a constructed edge v_iu_i as right constructed edge if v_1 belongs to $bd_c(v_iu_i)$, otherwise it is called a left constructed edge.

Definition 19 (Minimum set-covering problem). *Given a finite family C of sets $S_1, ..., S_n$, the problem is to determine the minimum cardinality $A \subseteq C$ such that*
$\cup_{i\in A} S_i = \cup_{j=1}^{n} S_n$

The problem of finding the minimum number of fans to cover P is same as the minimum set-covering problem, where every fan is a set and convex components are elements of the set.

Algorithm 1 Algo-Spiral(C)

1: Draw lines through every pair of vertices of P and compute all convex components $c_1, c_2, ..., c_m$ of P. Let $C = (c_1, c_2, ..., c_m)$, $N = (1, 2, ..., n)$ and $Q = \phi$.
2: For $1 \leqslant j \leqslant n$, construct the set F_j by adding those convex components of P that are totally visible from the vertex v_j.
3: Find $i \in N$ such that $|F_i| \geqslant |F_j|$ for all $j \in N$ and $i \neq j$.
4: Add i to Q and delete i from N.
5: For all $j \in N$, $F_j \leftarrow F_j - F_i$, and $C \leftarrow C - F_i$.
6: If $|C| \neq \phi$ then goto Step 3.
7: Output the set Q and Stop.

Theorem 25. *[Gho10] The approximation algorithm for the minimum vertex guard problem (VG) in a polygon P of n vertices computes solutions that are at most $O(logn)$ times the optimal. If P is a simple polygon, the approximation algorithm runs in $O(n^4)$ time. If P is a polygon with holes, the approximation algorithm runs in $O(n^5)$ time.*

Algorithm 2 Algo-Fans(C)

1: Draw lines through every pair of vertices of P and compute all convex components $c_1, c_2, ..., c_m$ of P. Let $C = (c_1, c_2, ..., c_m)$, $N = (1, 2, ..., n)$ and $Q = \phi$.
2: For $1 \leqslant j \leqslant n$, construct the set E_j by adding those convex components of P that are totally visible from the edge e_j.
3: Find $i \in N$ such that $|E_i| \geqslant |E_j|$ for all $j \in N$ and $i \neq j$.
4: Add i to Q and delete i from N.
5: For all $j \in N$, $E_j \leftarrow E_j - E_i$, and $C \leftarrow C - E_i$.
6: If $|C| \neq \phi$ then goto Step 3.
7: Output the set Q and Stop.

Theorem 26. *[Gho10] For the minimum edge guard problem (EG) in an n-sided polygon P, an approximate solution can be computed which is at most $O(logn)$ times the optimal. If P is a simple polygon, the approximation algorithm runs in $O(n^4)$ time. If P is a polygon with holes, the approximation algorithm runs in $O(n^5)$ time.*

Any set consisting of arbitrary chosen convex components may not form a fan as every fan consists of contiguous convex components. Therefore, constructing any example where the greedy algorithm takes $O(logn)$ times optimal does not seem to be possible.

5.3 O(logOPT)-Approximation [BM16]

Subir K Ghosh conjectured in 1987 the existence of a constant-factor approximation algorithm for Vertex and Edge Guard AGP, but there has been no result backing up the conjecture. First result in this direction was given in 2007, by Deshpande et al. [DKDS07] who gave a randomized pseudo-polynomial time $O(logOPT)$-approximation algorithm. The algorithm runs in time polynomial in n, where n is the number of walls and the spread. However in 2016, Miltzow et al. [BM16] found a bug in their results by giving a counterexample for their algorithm. Deriving from their ideas, Miltzow et al. gave an $O(logOPT)$-approximation algorithm for Point Guard AGP. This is the first correct randomized polynomial-time approximation algorithm for Point Guard AGP for simple polygons. The proof required 2 assumptions as follows:

(1) (integer vertex representation). Vertices are given by integers, represented in binary.
(2) (general position assumption). No three extensions meet in a point of P which is not a vertex and no three vertices are collinear where extension is a line obtained by extending the line-segment obtained by joining two points in both directions.

The basic idea derives from the general notion that Point Guard AGP can be seen as a geometric hitting set problem with an infinite set system. The only problem is that it still stands as an open problem to find a method the universe with an infinite set system to a finite number of elements. Assuming integer coordinates 1 gives useful lower bounds on distances between any two objects of interest that do not share a point as well as bounds the distance between any two vertices is by at least 1. The central technical component of the proof involves proving Global Visibility Containment Lemma 3. The main result is summarized as:

Theorem 27. *[BM16] Given $|OPT|$ as the cardinality of optimal solution, under Assumptions 1 and 2, there is a randomized $O(log|OPT|)$-approximation algorithm for Point Guard AGP for simple polygons that runs in polynomial time in the size of the input.*

Proof Idea: The main technical idea is to show the following lemma:

Lemma 3 (Global Visibility Containment). *Let P be some (not necessarily simple) polygon. Under Assumptions 1 and 2, it holds that there exists a grid* Γ *and a guard set* $S_{grid} \subseteq \Gamma$*, which sees the entire polygon and* $|S_{grid}| = O(|S|)$*, where S is an optimal guard set.*

From the above lemma it can be seen that the local visibility property holds for every point x that is far enough away from all extension lines. It can further be shown that it is impossible to be close to more than 2 extensions at the same time. □

For future directions what remains is to prove that Lemma 3 could be established without Assumption 2, but currently there are no results regarding the same and it remains as an open problem. Besides this, an important extension to this work can be done by proving $O(logOPT)$-approximation for polygon with holes. Another improvement could be obtaining super-constant inapproximability under standard complexity theoretic assumptions or improved approximation algorithms with a super-constant approximation factor.

5.4 6-Approximation for Weak Visibility Polygons [BGR15]

From have established that inapproximability of most of the variants of AGP proving that there is no c-approximation algorithm for simple polygons, for some constant c. For polygons with holes, they can even show that there is no $o(logn)$-approximation algorithm

Theorem 28. *[ESW01][SW] For weak visibility polygons with holes, there cannot exist a polynomial time algorithm for the Vertex Guard AGP with an approximation ratio better than* $(\frac{(1-\varepsilon)}{12})(lnn)$ *for every* $\varepsilon > 0$*, unless* $NP = P$*.*

We present a $6 - approximation$ algorithm given by Ghosh et al., which has running time $O(n^2)$, for vertex guarding polygons that are *weakly visible* from an edge uv and contain no holes. We begin by deifning a key structure used for the algorihtm as follows:

Definition 20 (Shortest Path Tree). *The* shortest path tree *of simple polygon P rooted at vertex s of P, denoted by* $SPT(s)$*, is defined as the union of* Euclidean shortest paths *from s to all the vertices of polygon P. Moreover, this* $SPT(s)$ *is a planar tree, rooted at s, which has n nodes, namely the vertices of P.*

For every vertex x of P, let $p_u(x)$ and $p_v(x)$ denote the parent of x in $SPT(u)$ and $SPT(v)$ respectively. In the same way, for every interior point y of P, let $p_u(y)$ and $p_v(y)$ denote the vertex of P next to y in the Euclidean shortest path to y from u and v respectively as depicted in Figure 16

Intuitively we use the concept of Euclidean shortest path trees from u and v for choosing vertices for placing guards. Let P be a simple polygon which is weakly visible from its edge uv. Suppose a guard is placed on every non-leaf vertex of $SPT(u)$ and $SPT(v)$. It is obvious that these guards see all points of P. However, the number of guards required may be very large compared to the size of an optimal guarding set. In order to reduce the number of guards, placing guards on every non-leaf vertex should be avoided. Let A be a subset of vertices of P. Let S_A denote the set which consists of the parents $p_u(z)$ and $p_v(z)$ of every vertex $z \in A$. Then, A should be chosen such that all vertices of P are visible from guards placed at vertices of S_A. Algorithm 3 presents a method for choosing A and S_A.

Lemma 4. *Any guard* $g \in OPT$ *that sees vertex z of P must lie on* $bd_c(p_u(z), p_v(z))$*.*

Lemma 5. *Let z be a vertex of P such that all vertices of* $bd_c(p_u(z), p_v(z))$ *are visible from* $p_u(z)$ *or* $p_v(z)$*. For every vertex x lying on* $bd_c(p_u(z), p_v(z))$*, if x sees a vertex q of P, then q must also be visible from* $p_u(z)$ *or* $p_v(z)$*.*

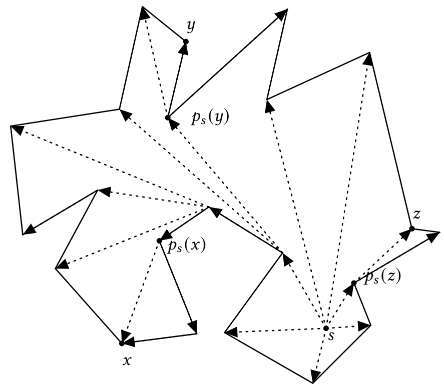


Fig. 16. Euclidean shortest path tree rooted at s.

Algorithm 3 Algo-SPT-Vertices(C)

Compute $SPT(u)$ and $SPT(v)$
Initialize all the vertices of P as unmarked
Initialize $A \leftarrow \phi$, $S_A \leftarrow \phi$ and $z \leftarrow u$
while $z \neq v$ **do**
    $z \leftarrow$ the vertex next to z clockwise on $bd_c(u, v)$
    if z is unmarked **then**
        $A \leftarrow A \cup \{z\}$ and $S_A \leftarrow S_A \cup \{p_u(z), p_v(z)\}$
        Place guards on $p_u(z)$ and $p_v(z)$
        Mark all vertices of P visible from $p_u(z)$ or $p_v(z)$
    end if
end while
return the guard set S_A

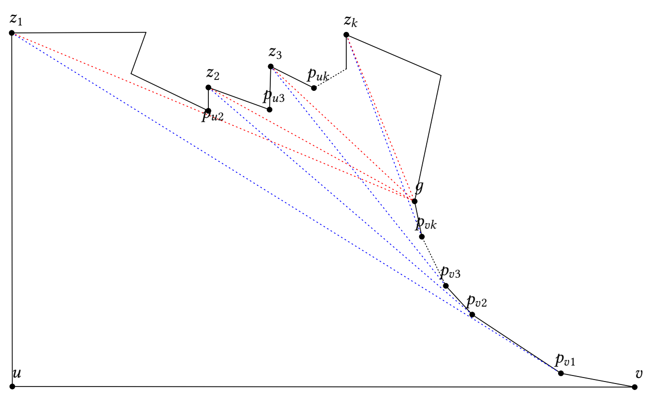


Fig. 17. Worst Case Example for Algorithm 3

Lemma 6. *If every vertex $z \in A$ is such that every vertex of $bd_c(p_u(z), p_v(z))$ is visible from $p_u(z)$ or $p_v(z)$, then $|A| \leqslant |OPT|$.*

Lemma 7. *If every vertex $z \in A$ is such that all vertices of $bd_c(p_u(z), p_v(z))$ are visible from $p_u(z)$ or $p_v(z)$, then $|S_A| \leqslant 2|OPT|$.*

Algorithm 4 Algo-SPT-Interior(C)

Compute $SPT(u)$ and $SPT(v)$
Initialize all the vertices of P as unmarked
Initialize $B \leftarrow \phi$, $S_B \leftarrow \phi$ and $z \leftarrow u$
while $\exists$ unmarked vertex in P **do**
    $z \leftarrow$ the first unmarked vertex on $bd_c(u, v)$ from z
    if every unmarked vertex of $bd_c(z, p_v(z))$ is visible from $p_u(z)$ or $p_v(z)$ **then**
        $B \leftarrow B \cup \{z\} \& S_B \leftarrow S_B \cup \{p_u(z), p_v(z)\}$
        Mark all vertices visible from $p_u(z)$ or $p_v(z)$
        $z \leftarrow p_v(z)$
    else
        $z' \leftarrow$ the first unmarked vertex on $bd_c(z, v)$
        while every unmarked vertex of $bd_c(p_u(z'), z')$ is visible from $p_u(z')$ or $p_v(z')$ **do**
            $z \leftarrow z'$
            $z' \leftarrow$ the first unmarked vertex on $bd_c(z', v)$
        end while
        $B \leftarrow B \cup \{z\} \& S_B \leftarrow S_B \cup \{p_u(z), p_v(z)\}$
        Mark all vertices visible from $p_u(z)$ or $p_v(z)$
        $y \leftarrow z$
        while $\exists$ an unmarked vertex on $bd_c(u, z)$ **do**
            $y \leftarrow$ first unmarked vertex on $bd_{cc}(p_u(y), u)$
            $B \leftarrow B \cup \{y\} \& S_B \leftarrow S_B \cup \{p_u(y), p_v(y)\}$
            Mark all vertices visible from $p_u(y)$ or $p_v(y)$
        end while
    end if
end while
return the guard set S_B

Lemma 8. $|B| \leqslant 2|OPT|$.

Lemma 9. $|S_B| \leqslant 4|OPT|$.

Proof. We have $|S_B| = 2|B|$. Also, by Lemma 5, $|B| \leqslant 2|OPT|$. So, $|S_B| = 2|B| \leqslant 4|OPT|$. While the guard set S_B is guaranteed to see all vertices of P, it may not always be true that all interior points of P are also visible from guards in S_B. As in Figure 18, while scanning $bd_c(u, v)$, Algorithm 4 places guards at $p_u(z)$ and $p_v(z)$ as all vertices of $bd_c(p_u(z), p_v(z))$ become visible from $p_u(z)$ or $p_v(z)$. Observe that in fact all vertices of P become visible from these two guards. But, $VP(p_u(z))$ has several left pockets and $VP(p_v(z))$ has several right pockets which intersect pairwise to create multiple invisible cells. In order to guard these invisible cells, a set S' of additional guards need to be placed. □

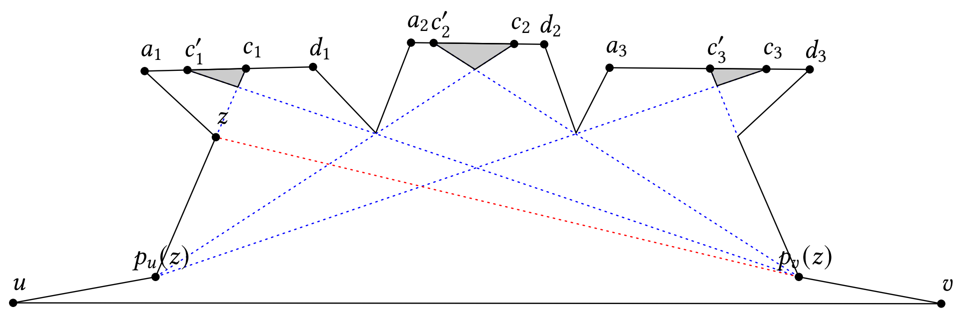


Fig. 18. Multiple invisible cells exist within the polygon that are not visible from the guards placed at $p_u(z)$ and $p_v(z)$.

Theorem 29. *There exists an algorithm with running time $O(n^2)$ that returns a guard set S' for guarding all interior points of P invisible from guards in S_B such that $|S'| \leqslant 2|OPT|$.*

Theorem 30. *There exists an algorithm with running time $O(n^2)$ that returns a guard set S for guarding all interior points of P such that $|S| \leqslant 6|OPT|$.*

5.5 More Results

(1) [KK11] J. King and D. Kirkpatrick, Improved Approximation for Guarding Simple Galleries from the Perimeter gave algorithm with Running time: $n^{O(1)}$, Approximation ratio of $O(loglogc_{opt})$.
(2) [BGM+14] S. K. Ghosh et al. Improved bounds for the conflict-free chromatic art gallery problem. Running time: $O(n^2)$. Chromatic guard number: $O(logn)$.

6 PARAMETERIZED ALGORITHMS

In the past 25 years, parameterized complexity have come up as one of the popular and fruit-bearing approaches for otherwise hard problem in theoretical computer science. The underlying notion lies in studying the problem in dependence with a natural parameter. In most of the cases, this dependence on the parameter is practical i.e. the parameter is usually small in practical-instances thereby allowing us to obtain near-polynomial running times producing the exact-optimal solution. It is relatively a new field in Analysis of Algorithms and has already rendered FPT algorithms for most of the hard problems including NP-Hard as well as APX-Hard problems. Thus we define the necessary the terminology required.

Definition 21 (Parameterized Problem). *[CFK+15] A Parameterized problem is a language $L \subseteq \Sigma^* \times N$, where Σ is a fixed, finite alphabet. For an instance $(x, k) \in \Sigma^* \times N$, k is called the parameter.*

This brings to think of possible algorithms and running times for the Parameterized Problems. We first define algorithms with running time $f(k)n^{O(1)}$, called as fixed-parameter algorithms, or FPT algorithms. Formally:

Definition 22 (Fixed Parameter Tractable(FPT) algorithms). *[CFK+15] A Parameterized problem $L \subset \Sigma^* \times N$ is called fixed parameter tractable (FPT) if there exists an algorithm A (called a fixed parameter algorithm), a computable function $f : N \longrightarrow N$, and a constant c such that, given $(x, k) \in \Sigma^* \times N$, the algorithm A correctly decides whether $(x, k) \in L$ in time bounded by $f(k).|(x, k)|^c$. The complexity class containing all fixed-parameter tractable problems is called FPT.*

Typically the goal in Parameterized algorithmics is to design FPT algorithms, trying to make both the $f(k)$ factor and the constant c; which is the power of n in running time; in the bound on the running time as small as possible. We further define another complexity with power of n as a function of input parameter as well as follows:

Definition 23 (Slice-wise polynomial (XP) algorithms). *[CFK+ 15] A Parameterized problem $L \subset \Sigma^* \times N$ is called slice-wise polynomial (XP) if there exists an algorithm A and two computable function $f, g : N \rightarrow N$ and given $(x, k) \in \Sigma^* \times N$, the algorithm A correctly decides whether $(x, k) \in L$ in time bounded by $f(k).|(x, k)|^{g(k)}$. The complexity class containing all slice-wise polynomial problems is called XP.*

FPT algorithms can be put in contrast with less efficient XP algorithms (for slice-wise polynomial), where the running time is of the form $f(k)n^{g(k)}$, for some computable functions $f,\ g$. It should be noted that there is a tremendous difference in the running times $f(k)n^{g(k)}$ and $f(k)n^{O(1)}$ ($f(k)n^c$). In Parameterized algorithms, k is simply a relevant secondary measurement that encapsulates some aspect of the input instance, be it the size of the solution sought after, or a number describing how "structured" the input instance is.

We study the problem in parameterized complexity considering the decision version of the problem where we check if there exists a solution set of guards of size k guarding the whole polygon. We can think of a very simple algorithm for the vertex guard variant to check if a solution of size k exists in a polygon with n vertices which runs in time $O(n^{k+2})$. This can be done by a simple brute force approach that checks all possible subsets of size k of the vertices. The first non-trivial result for point guard variant gives an algorithm with runtime $n^{O(k)}$ a set of k guards [EHP06]. It employs techniques from real algebraic geometry [BPR06]. Other than this, no major positive results are known for the classical variant (although numerous results have been known for variants of the problem). In the next subsection we prove lower bounds in parameterized complexity for AGP w.r.t the most natural parameter i.e. k, followed by a Fixed Parameter Tractable Algorithm for Vertex-Vertex AGP w.r.t number of reflex vertices.

6.1 ETH-based lower bounds

We present the first lower bounds for the parameterized AGP restricted to simple polygons. Here, the parameter is the optimal number k of guards to cover the polygon. A crucial assumption in proving almost every lower bound in parameterized complexity is based on Exponential Time Hypothesis (ETH) proposed by Russell Impagliazzo and Ramamohan Paturi. It can considered as a $P \neq NP$ equivalent of parameterized complexity and is stated as follows:

Conjecture 5. *[IP01, Exponential Time Hypothesis (ETH)] The* Exponential Time Hypothesis (ETH) *asserts that there is no* $2^{o(n)}$*-time algorithm for* $3-SAT$ *on instances with n variables.*

Using the ETH, lower bounds in runtime could be established for numerous number of problems which were "believed" to be hard were proven. Recently a breakthrough of AGP in parameterized complexity using ETH was proved by Bonnet et al. in the form of following negative result:

Theorem 31. *[BM20] [Parameterized hardness point guard] Classic Guard Art Gallery is not solvable in time* $f(k)\ n^{o(\frac{k}{logk})}$*, even on simple polygons, where n is the number of vertices of the polygon and* k *is the number of guards allowed, for any computable function* f*, unless the* ETH *fails.*

It can futher be extended to Vertex Guard variant of AGP as follows:

Theorem 32. *[BM20] [Parameterized hardness vertex guard] Vertex Guard Art Gallery is not solvable in time* $f(k)\ n^{o(\frac{k}{logk})}$*, even on simple polygons, where n is the number of vertices of the polygon and* k *is the number of guards allowed, for any computable function* f *, unless the ETH fails.*

In the next subsection we discuss the intuitive proof by Bonnet.
Proof Idea: As mentioned earlier, assuming ETH helped establish lower bounds on running times for numerous problems by a reduction from $3 - SAT$ (either direct or indirect). One such problem is *Structured 2-Track Hitting Set* (a variant of Hitting Set on 2-track interval graphs) which has its own lower bounds from ETH itself (follows from a reduction from variant of Multicolored Clique) as follows:

Theorem 33. *[BM20]* Structured 2-Track Hitting Set *is W [1]-hard. Furthermore it is not solvable in time* $f(k)|I|^{o(\frac{k}{logk})}$ *for any computable function f, unless the ETH fails.*

Major tool employed in the proof is the ability to encode Hitting Set on interval graphs. Assume that we have some fixed points $p_1, ..., p_n$ with increasing y-coordinates in the plane. We can build a pocket far enough to the right that can be seen only from $\{p_i, ..., p_j\}$ for any $1 \leqslant i < j \leqslant n$ as shown in Figure 19. Considering $I_1, ..., I_n$ be n intervals with endpoints $a_1, ..., a_{2n}$. Then, we construct $2n$ points $p_1, ..., p_{2n}$ representing $a_1, ..., a_{2n}$. Further, we construct one pocket far enough to the right for each interval as described above. This way, we reduce Hitting Set on interval graphs to a restricted version of the AGP. This observation is not so useful in itself since hitting set on interval graphs can be solved in polynomial time.
We can see positive results if we consider Hitting Set on 2-track interval graphs. The author constructs *linker gadgets* (as shown in Figure 20), which basically work as follows. We are given two set of points P and Q and a bijection σ between P and Q. The linker gadget is built in a way that it can be covered by two points (p, q) of $P \times Q$, if and only if $q = \sigma(p)$. The *Structured 2-Track Hitting Set problem* will be specifically designed so that the linker gadget is the main remaining ingredient to show hardness. □

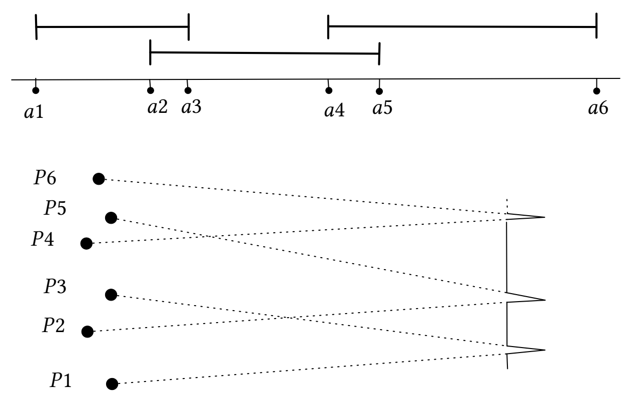


Fig. 19. Reduction from Hitting Set on interval graphs to a restricted version of the AGP.

6.2 FPT w.r.t reflex vertices

Very recently Lokshtanov et al. [AKL+20] gave a Fixed Parameter Tractable Algorithm running in time $r^{O(r^2)}n^{O(1)}$ for the Vertex-Vertex guard variant of the problem w.r.t r i.e. number of reflex vertices in the polygon. As defined in earlier, in Vertex-Vertex Guard (VVG) variant we have to guard all the vertices of the polygon placing guards at vertices only. That means that both the guard-positions and positions to be guarded are vertex set of the polygon. It is among very few positive results that concern optimal solutions for the problem and was published at 36th International Symposium on Computational Geometry (SoCG 2020).

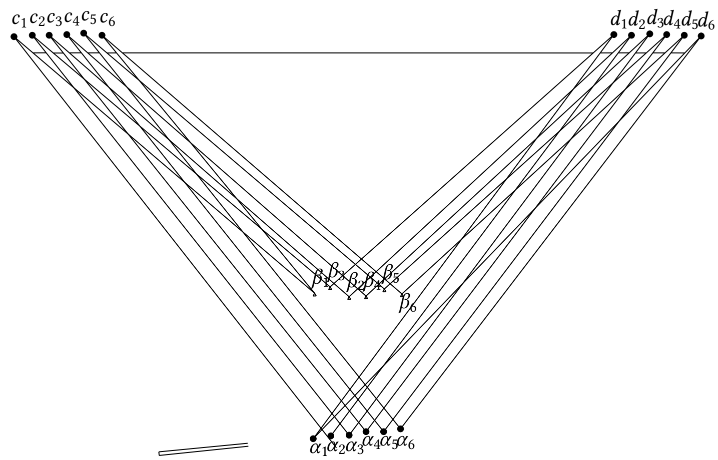


Fig. 20. Point Linker Gadget (using 3 weak linkers)

Definition 24 (Reflex Vertices). *A vertex V of a polygon is a reflex vertex if its internal angle is strictly greater than π.*

The first well-known proposition about algorithms for AGP w.r.t reflex vertices were proposed by Joseph et al in [A+90] stating the following conjecture :

Conjecture 6. *For any polygon P, the set of reflex vertices of P guards the set of all points within P.*

Later a similar open problem was pitched by Giannopoulos at Lorentz Center Workshop, 2016: "Guarding simple polygons has been recently shown to be W[1]-hard w.r.t. the number of (vertex or edge) guards. Is the problem FPT w.r.t. the number of reflex vertices of the polygon?". One of the initial observation was that the vertex-variant can be viewed as the classic Dominating Set problem in the visibility graph of a polygon.

Lokshtanov et al. [AKL+20] proved the following major theorem :

Theorem 34. *[AKL+20, Theorem 1] Vertex-Vertex (*VVG*)* AGP *is FPT parameterized by r, the number of reflex vertices. In particular, it admits an algorithm with running time $r^{O(r^2)}n^{O(1)}$.*

Proof Idea: The proof follows as a result of exponential-time reduction from the considered problem to Monotone 2-CSP. Thus first step involves designing an algorithm for Monotone 2-CSP running polynomial time of the input size. In Second step, the exponential time reduction is performed in two stages. (1) Reduction from Art Gallery to an annotated version of Art Gallery i.e. Structured Art Gallery. Intuitively, in Structured Art Gallery each convex region declares the number of guards it contains and also declares the number of guards required to guard the structured guard gallery in whole. This involves guessing the important visibility relations which are further elaborate. This blows up the reduction factor exponential as elaboration might take exponential number of guesses to exploit all the possibilities. (2) All the guessed instances are then reduced to instances of Monotone 2-CSP (Karp Reduction). The exponential-reduction can in fact be justified as it captures the "NP-hardness" of the problem. ◻

The above results can also be adapted to most general discrete annotated case of Art Gallery allowing G and C to be any subset of the vertex set of the polygon, which can include points where the interior angle is of 180 degrees. This gives the following result:

Theorem 35. *[AKL+20, Theorem 8] Vertex-Boundary (*VBG*) (as well as Boundary-Vertex(*BVG*)) Art Gallery is FPT parameterized by r, the number of reflex vertices. In particular, it admits an algorithm with running time $r^{O(r^2)} n^{O(1)}$.*

However, there have been no results on the Vertex-Point (VG) or Point-Vertex (PVG) variants of AGP and these still remain as open problems.

7 OTHER RESULTS

7.1 Practical Iterative Algorithm for PG AGP [TdRdS13]

AGP is an $\mathcal{NP}$-hard problem. We will now look at a practical iterative algorithm, which finds a sequence of decreasing upper bounds and increasing upper bounds to reach the optimal value. Note that this algorithm is for classical art gallery without holes, and for point guards. Let us define some more terms related to visibility. For any point p in the polygon P, we define *visibility polygon* $VP(p)$ as the set of all points of P visible to p. The edges of $VP(p)$ are called *visibility edges*. These definition can be extended to a finite set of points S. The region visible to this set S is simply the maximal connected region in union of the visibility polygons of individual points. Further, the visibility edges for the points in S partition the polygon P into a collection of convex polygonal faces called *Atomic Visibility Polygons(AVP)*.The arrangement defined by the visibility edges of points in S will be denoted as $Arr(S)$. The set of vertices of the AVPs of $Arr(S)$ will be denoted as $V(S)$. We now proceed with some more terms required to understand the approach.

For a general problem, the set of points to be covered P and the set of possible guard positions S are all points of the polygon, which is an infinite set. We will now look at some variations of the problem, which involve some finite sets. In Art Gallery Problem with *Fixed Guard Candidates (AGPFC)*, we are given a finite set of possible guard positions $C \subset S$, and we are required to find the minimum guards in C which cover the whole polygon. The second variant, *Art Gallery Problem with Witnesses (AGPW)*, we are given a finite set of points $W \subset P$, and we find the minimum number of guards to cover the set W. Note that covering W does not imply coverage of P. Remember that a polygon is called *witnessable* if there exists a finite witness set $W \subset P$ such that if any set of guards W, then they also cover P. The last variation which is a hybrid of the two, and brings sufficient amount of discretization to the problem is *Art Gallery Problem with Witnesses and Fixed Guard Candidates (AGPWFC)*, where both the points to be covered W and the potential guard positions C are finite. This problem can be cast as a *Set Cover Problem (SCP)*, where elements of W are to be covered using subsets of witness points visible to each guard in C. SCP is an $\mathcal{NP}$- hard problem, but many efficient solvers exist for them. This is utilized in the algorithm.

The working of the algorithm rests on the following two theorems :

Theorem 36. *Let D be a finite subset of points in P. Then, there exists an optimal solution for $AGPW(D)$ where each guard belongs to a AVP of $Arr(D)$.*

This theorem implies that we can obtain an optimal solution for $AGPW(D)$ by solving $AGPWFC(D, V(D))$. This problem, as mentioned earlier can be cast as *SCP*. Another important part is, as D is a subset of P, the answer is a lower bound for AGP. The second theorem says :

Theorem 37. *Let D and C be two finite subsets of P, so that C covers P. Assume that $G(D,C)$ is an optimal solution for $AGPWFC(D,C)$ and let $z(D,C) = |G(D,C)|$. If $G(D,C)$ covers P, then $G(D,C)$ is also an optimal solution for $AGPFC(C)$.*

We now briefly describe the algorithm. We initialize D to some set (the set of all vertices V works too. Strategies to construct the initial witness set has been experimented by the authors.) We initialize the lower bound LB to 0, upper bound UB to n and $G*$, the current known optimal guard set to V. Then, we iteratively solve $AGPW(D)$ (Using Theorem 36). It also gives us a new potential lower bound. We prepare the witness sets and candidate guard sets to obtain a new upper bound according to Theorem 37. We refer the readers to the original paper for more details of the algorithm, along with implementation details and computational experiments.

7.2 A Near-Minimal Witness Set [AÜ16]

Another interesting approach to look at the art gallery problem is to shrink the set of points required to be guarded (which is the entire polygon in the classical case) to a smaller subset of points. This set of points is called *Witness Set* of the polygon. A witness set W of a polygon P is defined as a set such that every set G guarding W also guards P. More formally :

Definition 25 (Witness Set). *A finite set $W \subseteq P$ is a Witness Set of P if for any set of point guards G in P, $W \subseteq \cup_{g \in G} VP(g)$ implies $\cup_{g \in G} VP(g) = P$, where $VP(v)$ is the visibility polygon of point v i.e. the set of points visible to v.*

There has been extensive research in this direction. We present recent results by Ayaz et al. who proposed an $O(n^4)$ runtime algorithm for finding a *near-minimal witness set* for a simple polygon.

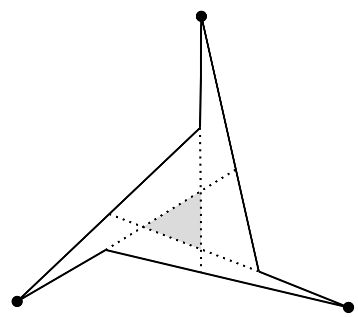


Fig. 21. Polygon with guard placemet guarding the boundary but not the entire area as the shaded triangle formed in the middle is not visible to these guards.

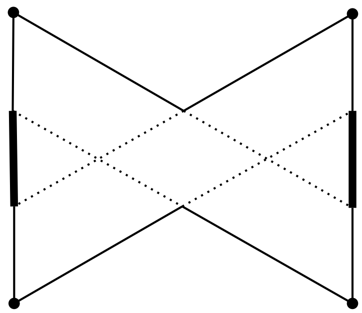
A question arises naturally whether guarding the boundary of polygon (denoted as δP) suffices to guard the whole polygon. Unfortunately the statement turns out to be false, a possible counterexample (as shown in Figure 21) of a polygon and a placement of guards such that the boundary is guarded but some area of polygon is left unguarded. Extending the concept of guarding the boundary first and then guarding interior was employed by Ayaz et al. to compute a minimal witness set. Intuitively the algorithm classifies points accordingly on the basis of visibility and placement of the polygon and then computes boundary-minimal witness set and interior-minimal witness set for the polygon.

More formally, a *minimal witness set* W_{min} for polygon P is a witness set such that there exists no proper subset of W_{min} that witnesses P. A polygon P is called *minimalizable polygon* if there exists a minimal witness set for it. Otherwise, P is called a *non-minimalizable polygon*. A *near-minimal witness set* is defined as follows:

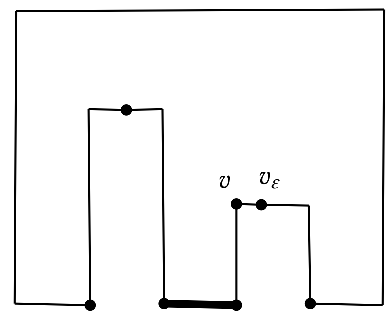
Definition 26 (Near-Minimal Witness Set). *A witness set W for a non-minimalizable polygon P is called near-minimal if it can be divided into two disjoint sets, W_{min} and W_ε such that W_ε consist of finitely many infinitesimally short line segment and removal of any element from W_{min} or W_ε makes W to violate the witnessing condition of W.*

We call each element of W_ε as an $\varepsilon - witness$

From Figure 22 it is evident that not all polygons admit a finite witness set of points. Thus the $O(n^4)$ algorithm computes witness sets of points, line segments and, if necessary, regions. And if the polygon has at least one



(a) A polygon wuth a minimal witness set of points and segments.



(b) Polygon with no minimal witness set but a near-minimal one, as it is necessary to include $\vec{vv^{+\varepsilon}}$.

Fig. 22. Polygons admitting no witness set of points

minimal witness set then the algorithm returns one such minimal witness set, else it returns a near-minimal witness set.

A point p is said to *see past left of* a reflex vertex v if the exterior of the polygon is on the left side of vector $\vec{pv}$ in the immediate neighbourhood of v (as shown in Figure 23). Analogous to this, we define p *seeing past right of* v. We say p *sees past* v if either p *sees past left or right* of v. We define *Cross Line* between two reflex vertices as line segment between a pair of reflex vertices that see past each other. *Visibility Kernel* $VK(p)$ is defined as the set of points that can see every point in visibility polygon of point p i.e. $VP(p)$. This definition can be extended to sets, i.e., given set of points S, $VK(S) = \cup_{p \in S} VK(p)$.

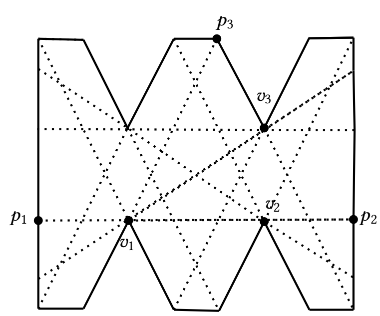


Fig. 23. Partition of a polygon boundary and inducing line segments and their extensions. p_1 sees past right v_1 and p_2 *sees past left of* v_1. v_1 *is of Type* 1, p_3 *is of Type* 2 and p_1 *is of Type* 3 anchor points. The line segment $\vec{v_1 v_3}$ is a cross line.

Firstly, points are classified with various labels. *Anchor points* are the points that subdivide the boundary of the input polygon. More formally, *Anchor points* include :(1) The vertices of the polygon, (2) For each reflex vertex, the boundary points where the extension of each of the two edges incident to it hit first, (3) For every pair of reflex vertices that see past each other, the boundary points where the two extensions of the line segment between them hit first (See Figure 23). *Anchor Edge* is defined as a line segment joining two consecutive anchor points. A point is labelled as an *Ordinary Point* iff $p \in P \setminus VK(\delta P)$ and p does not lie on a cross line. Points are further classified in five types according to the vertices they see past left and/or right. For the sake of simplicity we just mention the types *Type Z, Type L, Type R, Type D, Type N* and refer the reader to the main paper [AÜ16] for further in depth information.

We now highlight some of the results that serve as building blocks for the main algorithm.

Lemma 10. *[AÜ16] Every point on an anchor edge sees past left and right the same set of reflex vertices (due to Type 2 anchor points). Also these points (partially) sees the same set of edges (due to Type 3 anchor points). Moreover the leftmost and the rightmost reflex vertices a point sees past is the same for all points within an anchor edge.*

Lemma 11. *[AÜ16] An ordinary point p can only witness itself, i.e., $VK(p) = \{p\}$. Moreover p is present in every witness set of P.*

Lemma 12. *[AÜ16] Let p and q be distinct points. If $VK(p) \subset VK(q)$, then p cannot be in any minimal witness set.*

Lemma 13. *[AÜ16] If a point p is in $VK(\delta P) \setminus \delta P$ then p cannot be in any minimal witness set.*

Lemma 14. *[AÜ16] If a point p on δP is not witnessed by another point on δP, then p has to be in every minimal witness set.*

Lemma 15. *[AÜ16] A minimal witness set consist of a set of boundary elements, all ordinary points, and at least one point on each cross line that is in $P \setminus VK(\delta P)$.*

We now define the processing in the main algorithm briefly. As discussed earlier, in the first step all anchor points and the corresponding edges are searched by using a standard linear time visibility algorithm on each vertex. The next steps compute a set formed by subdivision of the polygon based on an arrangement of three types of line segments (described in the main paper). The next step computes elements of a near-minimal witness set on the boundary i.e. δP followed by finding the cells of a near-minimal witness set in the interior of polygon P (interior is all points in $P \setminus \delta P$).

The returned set W is minimal if it contains no $\varepsilon - witness$ element. Otherwise, it returns W which is a near-minimal solution to the non-minimizable polygon.

8 CONCLUSION AND FUTURE DIRECTION

AGP is one of the most studied problem in computational geometry. We attempted to cover the key results of the problem considering the major variants w.r.t the guard set as well as the type of polygon. During the study we encountered some problems which are potentially open for AGP and a few possible extensions from the work we mentioned. These are summarized as follows:

(1) Bounding the number of variables which are sufficient as well as necessary to for encoding AGP, say a bound of $\omega(k)$ variables.
(2) Super-constant inapproximability or improved approximation algorithms.
(3) $O(logOPT)$-approximation algorithm running in polynomial time for polygon with holes.
(4) FPT w.r.t number of reflex vertices for Vertex-Point (VG) or Point-Vertex (PVG) variants of AGP still remain as open problems.
(5) Prove that Lemma 3 could be established without Assumption 2 thereby giving $O(logOPT)$-approximation without Assumption 2.
(6) The approach used in section Section 6.2 can be proven fruitful to resolve the parameterized complexity of other problems of discrete geometric flavour. The problem still open with respect to parameterized algorithms for parameters like Tree Width, Clique Width etc.
(7) Determining a tight bound for vertex in classical art gallery with holes (as mentioned in Section 4.1.1). Rourke's result is a loose upper bound, while Shermer's conjecture is still open for general number of vertices and holes.

(8) Proving the 4 : 3 ratio between the extreme number of points and mobile guards for art galleries provided by both simple polygons and simple orthogonal polygons.

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